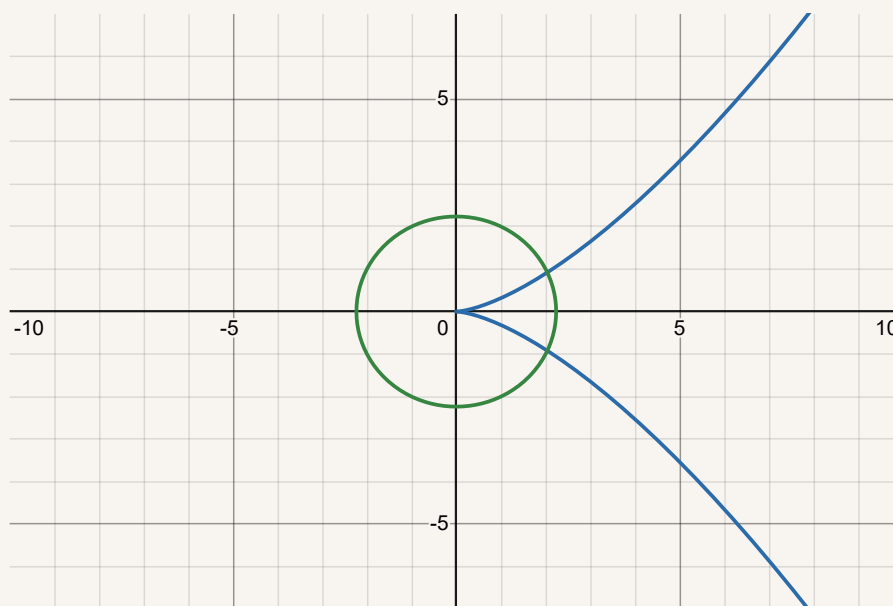


Spring, summer '25 and beyond:

Geometry of flows
and analysis on manifolds



Short link: <https://dub.sh/spring25>

Typeset by [Rupadarshi Ray](#)

while throwing ideas with my peers [Manan Jain](#), Pahul Arora and Naman Narang
at [IISER Mohali](#)

Last updated: 2:25 AM - August 09, 2025

timeline



'25.

The spring semester in IISER Mohali runs from January to early May:



ended 94 days ago

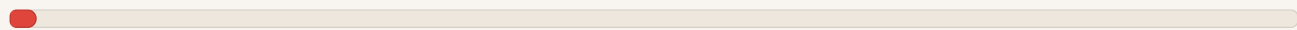
Then we have summer:

Summer holiday's will start from Wed 7 May, '25 and end on Mon 4 Aug, '25.

ended 5 days ago

And then the *thesis year begins* (for 3/4 of us above, anyways):

MS thesis period will start from Mon 4 Aug, '25 and end on Sat 4 Apr, '26 tentatively.



238 days left in MS thesis year of 244 days

2.09% complete

notebooks

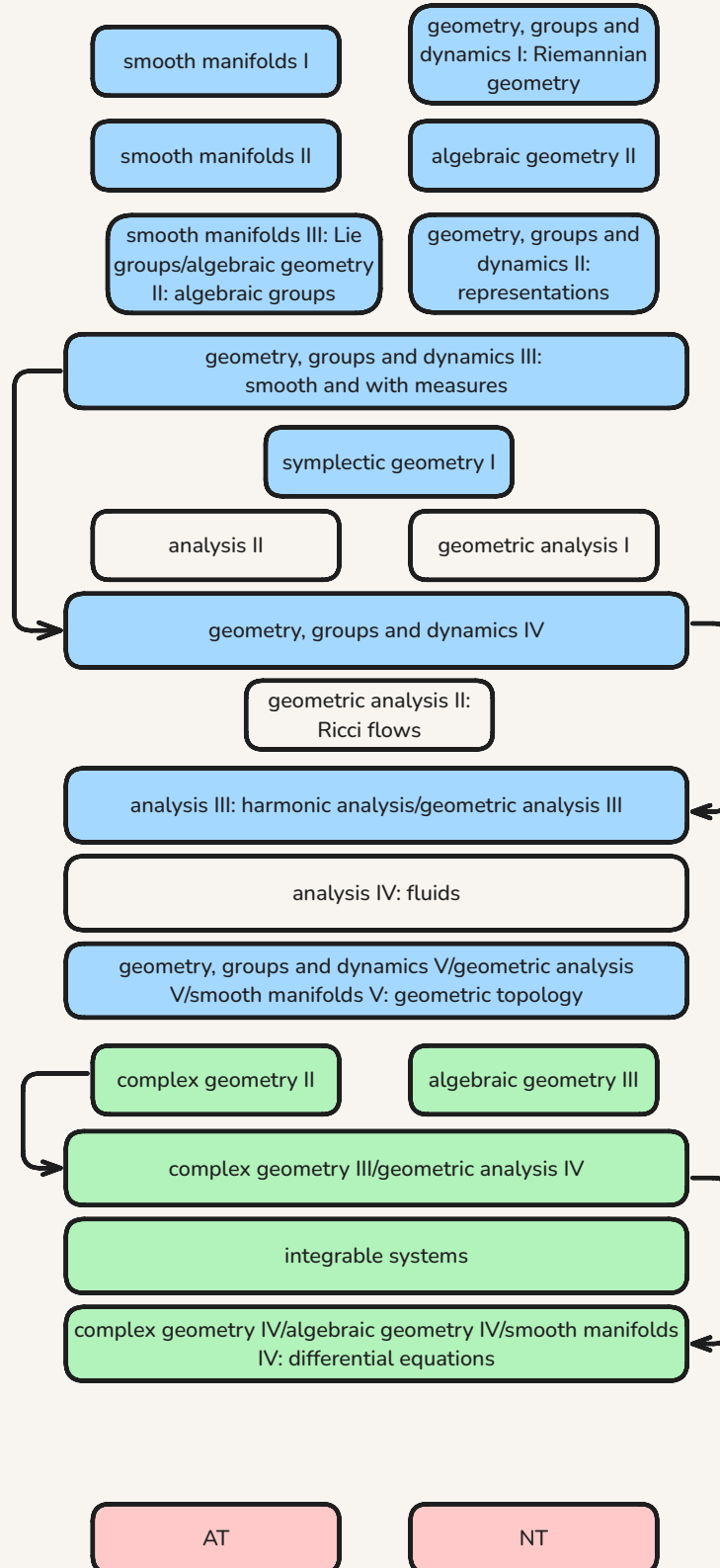
This piece of notebook is a part of a *bigger* set of notebooks

notebooks

#notes

- the *blue* notebook on *geometry, groups and dynamics* is [available here](#)
 - a subset consisting of the notes on the *Summer School on Rigidity of Discrete Groups, June 30 – July 4, 2025 at IISER Mohali* is at <https://dub.sh/rigidity>
 - a subset consisting of notes on *coadjoint orbits and quantization of Hamiltonian actions* is at <https://dub.sh/qofhamil>
 - a subset containing notes on *introductory Hamiltonian toric actions* <https://dub.sh/symplectictoric>
- the notebook on *complex geometry I: complex 1-manifolds* is available in the short link <https://dub.sh/riemann>
- the *white* notebook (analysis)
- the *green* notebook (CG/AG, integrable systems)
- the *epilogue*

with topics:



Warning

Every note in what follows is **complete** and **correct** *asymptotically*...

contents

- **smooth manifolds I**
 - [manifolds](#)
 - [Counting compact topological manifolds upto homeomorphism](#)
 - [Mikhail Gromov - What is a Manifold](#)
- **Riemannian geometry I**
 - [Riemannian connection](#)

- [Parallel transport on a curve in a semi-Riemannian manifold](#)
- [Curvature of a Riemannian manifold](#)
- [List of Riemannian manifolds](#)
- **analysis I**
 - [integration by parts and pieces](#)
 - [PDEs and orthonormal basis of \$L^2\(M\)\$](#)
 - [IMPA 2019 Functional analysis](#)
 - [Following Evans' book on PDEs](#)
 - [Papa Rudin](#)
 - [M E Taylor - PDEs](#)
 - [Larry Guth - MIT Differential Analysis 2](#)
 - [2503 CMI - GMT](#)
 - [modern Fourier analysis](#)
- **contents: complex geometry I: complex 1-manifolds**
AKA [Riemann surfaces](#)
 - [The complex tangent space on \$\mathbb{C}\$](#)
 - [Complex differential forms on \$\mathbb{C}\$](#)
 - [Integral of differential forms on \$\mathbb{C}\$](#)
 - [Holomorphic functions on \$\mathbb{C}\$](#)
 - [Holomorphic mapping onto disk, Riemann mapping theorem](#)
 - [2506 Seminar on Riemann surfaces](#)
 - [Analytic continuation and maximal domain of holomorphic functions](#)
 - [Global holomorphic functions](#)
 - $\sum_{n \geq 0} z^{n!}$
 - [Constructing Riemann surfaces](#)
 - [The usual definition of Riemann surfaces](#)
 - [Holomorphic functions on a complex 1-manifold](#)
 - [Differential forms on complex 1-manifolds](#)
 - [Cohomology groups of Riemann surfaces](#)
 - [Concept of a Riemann surface: usual definition vs as a ringed space](#)
 - [Classification of Riemann surfaces](#)
 - [Constructing meromorphic functions on Riemann surfaces](#)
 - [periods of elliptic curves](#)
 - [book readings](#)
 - [Simon Donaldson - Riemann surfaces](#)
- **geometric analysis I**
 - [Ancient solutions to geometric flows](#)
 - [Geometric analysis \(Julie Rowlett\)](#)
 - [Seminars on Inverse Problems, Julie Rowlett, June 7, 2022](#)
 - [Masoud Khalkhali - Spectral geometry](#)
 - [questions on spacetimes](#)
- **symplectic geometry I/dynamics I: symplectic dynamics**
 - [2502 GSG symplectic geometry and Hamiltonian flows](#)
 - [Corinna Ulcigrai - Chaotic Properties of Area Preserving Flows](#)

- [IMPA19 School of Symplectic Topology](#)
 - [Denis Auroux - Fukaya categories and mirror symmetry](#)
 - [Hofer, Zehnder - Symplectic Invariants and Hamiltonian Dynamics](#)
-

• algebraic geometry I

- [Joe Harris - Algebraic Geometry: a first course](#)
- [varieties](#)
- [schemes](#)

• groups I: Lie groups, locally compact groups, algebraic groups and their representations

Representation theory of (finite, Lie) groups and Lie algebras *along with any* (classical algebraic projective, symplectic) geometry that arises.

- Reading [Bump?](#)
 - [Fulton & Harris - Representation theory](#)
 - [continuous representations of locally compact groups](#)
 - complex semi-simple
 - $\mathfrak{sl}(2, \mathbb{C})$
 - [FinVec \$_{\mathbb{C}}\$ irreducible representations of \$\mathfrak{sl}\(2, \mathbb{C}\)\$ and \$SL_{\mathbb{C}}\(2\)\$](#)
 - $2 \otimes 3 = 5 \oplus 3 \oplus 1$ for [FinVec \$_{\mathbb{C}}\$ irreps of \$\mathfrak{sl}\(2, \mathbb{C}\)\$](#)
 - $\mathbf{Sym}^2 \mathbf{Sym}^2(\mathbb{C}^2) \cong_{SL_{\mathbb{C}}(2)} \mathbf{Sym}^0(\mathbb{C}^2) \oplus \mathbf{Sym}^4(\mathbb{C}^2)$
 - compact groups
 - [M S Raghunathan.cpt Lie and reps](#)
 - orbit method
 - [Ad* orbits are symplectic manifolds](#)
 - Nilpotent groups
 - orbit method/geometric quantization and Borel-Weil theorem
 - algebraic groups
 - [IMSC-Linear algebraic groups](#)
 - [GSG talk on On Character Variety of Anosov Representations by Tathagata Nayak](#)
-

• dynamics II

- [ICTP18 Dynamics](#)
- [McMullen - Ergodic theory, geometry and dynamics](#)
- [Einsiedler and Ward - Ergodic Theory](#)
- [Pesin B - Introduction to smooth ergodic theory \(2013\).pdf](#)
- [Anosov flows](#)
- [Uri Bader - A course in linear groups and ergodic theory](#)

• analysis II: harmonic, microlocal, semiclassical

- [A Pirkovskii - Harmonic Analysis and Banach Algebras](#)
- A talk I attended in Mohali: [Utsav Dewan - Boundary exceptional sets for radial limits of positive superharmonic functions on Harmonic manifolds](#)
- [The three-dimensional Keakeya conjecture, after Wang and Zahl | What's new](#)

- [Harmonic analysis on homogeneous spaces](#)
 - [Sigurdur Helgason - Groups and geometric analysis](#)
-

- **homogeneous spaces**

- [0](#)
- [REF homogeneous spaces](#)

- **Riemannian geometry II/groups II: symmetric spaces**

- [【ETHZ】Symmetric Spaces 对称空间_哔哩哔哩_bilibili](#) gone! 😞

- **geometry, groups, rigidity and dynamics**

- courses
 - [TIFR-Geometry, groups and dynamics \(2017\)](#)
 - [Kathryn Mann - Dynamics of Group Actions](#)
 - [Fields Institute - Geometry, Arithmetic, and Dynamics of Discrete Groups](#)
 - [Programa de Doutorado: Lie Groups, Representation Theory and Symmetric Spaces - YouTube](#)
 - [10ICTS Geometry Topology and Dynamics in Negative Curvature](#)
 - [IHEs 23 Homogeneous Dynamics and Geometry in Higher-Rank Lie Groups](#)
 - [Ragunathan - Discrete subgroups of Lie groups](#)
 - [Rodriguez Hertz - Rigidity](#)
- schools I've attended
 - [Summer School on Rigidity of Discrete Groups, June 30 – July 4, 2025 at IISER Mohali](#)
- talks, articles
 - [Karen Butt - Entropies of negatively curved surfaces](#)
 - [Fanny Kassel - Geometric structures and representations of discrete groups](#)
 - [Sandip Singh - Arithmeticity and Thinness of Hypergeometric Groups](#)
- books
 - [Amie Wilkinson - Geometry, Dynamics, and Rigidity](#)
 - [Dave Witte Morris - Arithmetic groups](#)
 - [deductivepress.ca/dmorris/books/ZimmerCBMS.pdf](#)
 - [Robert J. Zimmer - Ergodic Theory and Semisimple Groups \(1984, Birkhäuser\).pdf](#)
 - [Werner Ballmann, Mikhael Gromov, Viktor Schroeder - Manifolds of Nonpositive Curvature \(1985, Birkhäuser\).pdf](#)
- **geometric topology, low dimensional topology and more**
 - Mohali
 - [240925 GSG orderability in 3-manifold groups](#)
 - [2501 GSG liftable mapping class groups](#)
 - [Young Mathematicians' Symposium 2025, IISER Mohali](#)
 - [Anushree Das - Volume growth of complete Riemannian manifolds under suitable curvature conditions](#)
 - notes and lectures
 - [J. Aramayona - MCG and infinite MCG \(Part 2\)](#)

The proof, Part I: A volume preserving conjugacy

We begin the proof of Otal's theorem. We start with:

Proposition 3.1. *The space of negatively curved metrics on a closed surface S is path connected.*

IDEA OF PROOF. Hamilton's Ricci flow on surfaces preserves negative curvature and converges to a metric of constant negative curvature [20]. Given two metrics on a surface, flow both by Ricci flow to conformal metrics, and if necessary scale the metrics to have constant negative curvature -1 . Then both flowed metrics belong to Teichmüller space. Since Teichmüller space is connected, the two metrics can then be connected by a path through metrics of constant negative curvature.

◊

[1]

•

- [Inst Fourier - Low dim topology](#)

- broad topics and courses

- [3-manifolds](#)
- [Foliations](#)
- [MIT18.96x geometry](#)

We add more topics which we shall *not* pursue more.

- **geometric analysis II: Ricci flows**

- [Following Chow Lu Ni - Hamilton's Ricci flow](#)
- [Gerhard Huisken - Ricci flow](#)
- [Einstein metrics and Ricci solitons](#)
- [Recent developments in Ricci flows](#) <https://arxiv.org/pdf/2102.12615>

- **analysis IV: rigids, gases, fluids**

- points
 - [Potential flows](#)
- rigid
 - [Configuration space of \$N\$ rigid points in \$\mathbb{R}^3\$](#)
 - [Riemannian mechanical manifold of a rigid body in \$\mathbb{R}^3\$ with a point fixed](#)
- fluid
 - [IMPA Hidden symmetries in fluid dynamics 2024](#)
 - [Maria Colombo - Flows of Irregular Vector Fields in Fluid Dynamics](#)
 - [Boris Khesin - Geometric fluid dynamics](#)
 - Mathematical Turbulence [Programa de Doutorado: Tópicos em Matemática da Turbulência \(2022\)](#)
- gases
 - [Carlo Cercignani, Reinhard Illner, Mario Pulvirenti - The Mathematical Theory of Dilute Gases-Springer-Verlag New York \(1994\).pdf](#)
 - [Laure Saint-Raymond - Hard Sphere Dynamics in the Low Density Limit](#)

More stuff we shall *only* pursue later

- **complex geometry II**

- [Pierre Albin - Complex Algebraic Geometry](#)
- [IMPA Hodge theory](#)
- [John Milnor - Singular Points of Complex Hypersurfaces](#)

- [Lecture 1 : Singular Levi-flat hypersurfaces by Jiri Lebl](#)
- **algebraic geometry III**
 - [2503 GSG Rational curves and a conjecture of Drinfeld](#)
 - [March 17-19, 2025 - Geometric Aspects of Algebraic Varieties, IISER Mohali](#)
 - [Kabeer MR - Metric techniques for triangulated categories](#)
 - *Intersection theory*
 - [Intersection Theory \(1 of 5\) - YouTube](#) [Intersection theory in algebraic geometry - lccs \(columbia.edu\)](#)
- **complex geometry III**
 - [Complex Geometry by Hans-Joachim Hein](#)
 - [Harish Seshadri - Comparison theorems in Kahler geometry](#)
 - GIT
 - [GIT → moduli of vector bundles, stability of vector bundles → Einstein-Kahler metrics](#)
 - [toric](#)
- **integrable systems**
 - [Nicolai Reshetikhin - Integrable systems and representation theory: geometry, algebra and analysis](#)
 - *Integrable PDEs*
 - [KdV equation](#)
 - *Algebraic integrable systems*
 - [BIMSA - Integrable Systems and Algebraic Geometry - Beijing Summer Workshop in Mathematics and Mathematical Physics \(June 24 – July 5, 2024\)](#)
- **complex geometry IV/algebraic geometry IV/smooth manifolds IV:**
differential equations
 - [Loray - Painlevé equations and isomonodromic deformations](#)
 - [Henryk - The Monodromy Group](#)
 - [GADEPs focused conference: Abelian and iterated integrals and Hilbert 16th problem \(2022\)](#)
 - [NCTS Short Course on Riemann Hilbert Method in Integrable Systems](#)
 - [algebraic study of ODEs](#)
 - [Peter J. Olver - Applications of Lie Groups to Differential Equations -Springer \(2000\).pdf](#)

We now continue... with some *fancy lore*

- **smooth manifolds II**
 - *hatcher - AT*
 - [homotopy extension property of cone](#)
 - [Jet Nestruev - Smooth Manifolds and Observables](#)
 - *Global calculus*
 - *Differential forms*
 - *Vector bundles*
 - *Differential topology*
 - *Differential cohomology*
 - [Gregory Moore - Differential Cohomology, TQFTs and beyond](#)
- **(co)homology ↔ Lagrangian field theories ↔ integrable systems**
 - paper reading [Andrew Neitzke - BPS](#)

- [Sergei Gukov - Homological algebra of knots and BPS states](#)
- [Symplectic Duality and Geometric Langlands Duality - TIB AV-Portal](#)
- Following [A Guest - From Quantum Cohomology to Integrable Systems](#)
- [Donaldson](#)
- **arithmetic and topology**
 - [Number fields](#)
 - [knots and number fields](#)
- **mirror symmetry**
 - [Calabi-Yau manifolds](#)
 - [Mark - what is mirror symmetry](#)

-
- [endnotes](#)
 - [philosophy of mathematics, 16 march '25 edition](#)
-

1. math.uchicago.edu/~wilkinso/papers/PCMI-Wilkinson.pdf?utm_source=chatgpt.com#page=29.24 ↩

manifolds

The only division rings $(F, +, \times)$ which are also smooth $\mathbb{R}$ -manifolds (compatible) possible are $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. [1]

- $\mathbb{R}$ -manifolds

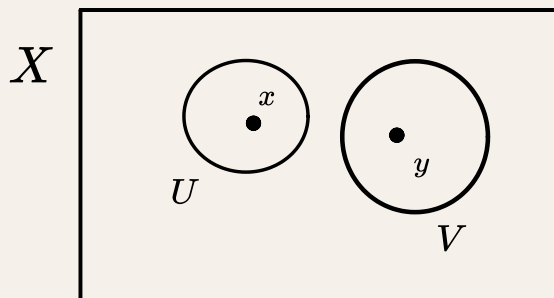
Definition. Topological manifold

A **topological space** is a real topological n -manifold or $(\mathbb{R}^n, \mathcal{C}^0)$ -**manifold** for a $n \in \mathbb{Z}^+$ (or topological n -manifold) if

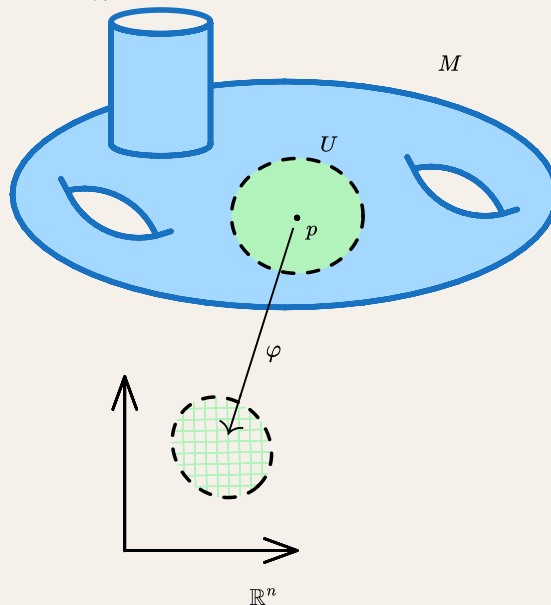
- M is Hausdorff topological space

Definition. T_2 topological space or Hausdorff space

A **topological space** is said to be **T_2 or Hausdorff** if for any pair $x, y \in X$, $\exists U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$.



- M is **second countable**
- (locally homeomorphic to $(\mathbb{R}^n, \mathcal{T}_{\text{std}})$) for any $p \in M$, there exists



- an open subset $U \subseteq M$ with $p \in U$
- open subset $V \subseteq \mathbb{R}^n$
- and a homeomorphism $\varphi : U \rightarrow V$

- PL manifold
- $\mathcal{C}^k$ -manifold
- **smooth $\mathbb{R}$ -manifold**

Definition. Homogeneous manifold

A **transitive smooth Lie group action** on a **smooth $\mathbb{R}$ -manifold** is called a homogeneous G -manifold.

Definition. **Spin structure on a Riemannian vector bundle**

Let

$$E \rightarrow M$$

be a oriented Riemannian $\mathbb{R}^n$ -vector bundle over M and let $P_{SO}(E)$ be its bundle of oriented orthonormal frames.

- For $n \geq 3$, we have the universal covering homomorphism

$$\text{Spin}_n : \text{Spin}(n) \rightarrow \text{SO}(n)$$

with kernel $\{I, -I\} \cong \mathbb{Z}_2$. A **spin structure** on E is a principal $\text{Spin}(n)$ -bundle $P_{\text{Spin}}(E)$ together with a 2-sheeted covering

$$\zeta : P_{\text{Spin}}(E) \rightarrow P_{SO}(E)$$

such that $\zeta(pg) = \zeta(p)\text{Spin}_n(g)$ for all $p \in P_{\text{Spin}}(E)$ and $g \in \text{Spin}(n)$

- For $n = 2$

Definition. A **symplectic manifold** (M, ω) is a **smooth manifold** M with a closed non-degenerate 2-form ω .

Definition. **Toric manifolds** (M, ω, μ)

A **symplectic manifold** (M, ω) is a **toric manifold** if it is equipped with a faithful **Hamiltonian Lie group action** of a torus

$$T^{\dim M/2} \curvearrowright (M, \omega)$$

with the moment map

$$\mu : M \rightarrow \mathbb{R}^{\dim M/2}$$

The image of the moment map $\mu(M) \subseteq \mathbb{R}^{\dim M/2}$ is called the **Newton polytope** of (M, ω, μ) .

Definition. **Lie group**

A group object in **Man**.

Definition. **Riemannian metric on a smooth manifold**

A **Riemannian metric** g on a smooth manifold M is a section of $\text{Sym}^2 T^*M$ which is positive definite.

Definition. **Sub-Riemannian manifolds**

A *sub-Riemannian manifold* is a triple (M, Δ, g) , where M is a differentiable manifold, Δ is a bracket-generating distribution, and g is a smooth section of positive-definite quadratic forms on Δ . In fact, the map g can be considered as the restriction to Δ of a Riemannian metric tensor on the manifold M . A curve γ on M is called *admissible*, or *horizontal*, with respect to Δ if it is absolutely continuous and $\dot{\gamma}(t) \in \Delta_{\gamma(t)}$ for almost every t . Then the *sub-Riemannian distance* is defined by the same formula $(\star)$. More generally, if the length comes from a smoothly varying norm, then the distance is called *Carnot-Carathéodory metric* or *sub-Finsler*. Most of the previously mentioned results on the Heisenberg group will be valid for every Carnot-Carathéodory space.

Definition. Finsler manifold

Definition 2.2.9. A *continuously varying norm* on a differentiable manifold M is a continuous function from TM to $[0, \infty)$ usually denoted by $\|\cdot\|$

$$\|\cdot\| : TM \rightarrow [0, \infty), \quad X \in TM \mapsto \|X\|,$$

with the property that for all $p \in M$ the restriction of $\|\cdot\|$ to $T_p M$ is a symmetric norm, i.e.,

1. $\|\lambda X\| = |\lambda| \|X\|, \forall X \in TM, \forall \lambda \in \mathbb{R};$
2. $\|X + Y\| \leq \|X\| + \|Y\|, \forall p \in M \text{ and } \forall X, Y \in T_p M;$
3. $\|X\| = 0 \Rightarrow X = 0.$

Definition 2.2.10. In this text, we say that a *Finsler manifold* is a pair $(M, \|\cdot\|)$ where M is a differentiable manifold and $\|\cdot\|$ is a continuously varying norm on M , in the sense of Definition 2.2.9. In this case, the function $\|\cdot\|$ is also called *Finsler structure*.

Definition. Poisson manifold

A **Poisson manifold** (M, B) is a smooth manifold M with a lie bracket

$$B : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$$

on the $\mathbb{R}$ -vector space $\mathcal{C}^\infty(M)$ which satisfies the Leibniz identity

$$f, g, h \in \mathcal{C}^\infty(M) \implies B\{f, gh\} = B\{f, g\}h + gB\{f, h\}$$

Definition. Multisymplectic manifold

A **multisymplectic manifold** (M, ω) of degree $k \geq 1$, or a **k -plectic manifold**, is a **smooth $\mathbb{R}$ -manifold** with a closed differential $k + 1$ -form ω which is non-degenerate, that is

$$\begin{aligned} \text{Vec}(M) &\rightarrow \Omega^k(M) \\ X &\mapsto \iota_X \omega \end{aligned}$$

is injective.

- [sett.Man.R J](#) *Almost complex manifold*
 - [sett.Man.R triple](#)
 - $\mathbb{C}$ -manifold [complex manifold](#)
 - [sett.Man.C group](#)
 - [sett.Man.C Kahler](#)
 - manifolds based on topological vector spaces over $\mathbb{R}, \mathbb{C}$
 - Banach manifold
 - Frechet manifold
 - $\mathbb{H}$ -manifold [Quaternionic manifold - Wikipedia](#) ^[2]
 - $\mathbb{O}$ -manifold
-

- *Locally ringed spaces* are far too general?
-

- analytic k -manifolds where k is a [field](#) with a absolute value which is metric complete. ^[3] apart from $k = \mathbb{R}, \mathbb{C}$
 - $\mathbb{Q}_p$
 - $\mathbb{Q}$ -manifold?
-

- [Supermanifold - Wikipedia](#)
-

1. [Are there only two smooth manifolds with field structure: real numbers and complex numbers? - MathOverflow](#) ↩
2. [dg.differential geometry - Is there a reasonable definition of an octonionic manifold? - MathOverflow](#) ↩
3. [3.dvi \(ucla.edu\)](#) ↩

Counting compact topological manifolds upto homeomorphism

[1]

without boundary

Suppose there are not countably many of them, then there are an uncountable number of them for some dimension say n . Let

$$\{M_\alpha \mid \alpha \in A\}$$

be such a collection.

For each α we find a collection of imbeddings for $1 \leq j \leq k_\alpha$

$$h_{\alpha j} : B(2) \subset \mathbb{R}^n \rightarrow M_\alpha$$

such that

$$\{h_{\alpha j}(B(1)) \mid 1 \leq j \leq k_\alpha\} \text{ covers } M_\alpha$$

We may assume

- $k_\alpha = k$ for all α by choosing a uncountable subcollection
- $h_{\alpha j} \mid_{B(1)}$ can be extended to an imbedding of $B(k+1) \rightarrow M_\alpha$
- $M_\alpha \subset \mathbb{R}^{2n+1}$

$$\begin{aligned} \epsilon_{\alpha j m} &:= d(h_{\alpha j}(B(m)), M_\alpha - h_{\alpha j}(B(m+1))) \\ \epsilon_\alpha &:= \min_{j,m} \{\epsilon_{\alpha j m}\} \end{aligned}$$

We again assume $\exists \epsilon > 0$ such that $\epsilon_\alpha > \epsilon$ for all $\alpha \in A$ by choosing a subcollection.

Each M_α determines an imbedding

$$\begin{aligned} g_\alpha : B(k+1) &\rightarrow (\mathbb{R}^{2n+1})^k \\ g_\alpha(x) &:= (h_{\alpha,1}(x), \dots, h_{\alpha,k}(x)) \end{aligned}$$

The set of all such imbeddings $\{g_\alpha \mid \alpha \in A\}$ under the uniform metric

$$d_u(g_\alpha, g_\beta) := \max_{x \in B(k+1)} d(g_\alpha(x), g_\beta(x))$$

is a separable metric.

Hence, g_{α_0} is a limit point of a sequence of distinct imbeddings $g_{x_1}, \dots$

Proposition: $M_{\alpha_0} \cong M_{\alpha_i}$ for i sufficiently large. Furthermore, this homeomorphism can be taken close to $\iota : M_{\alpha_0} \hookrightarrow \mathbb{R}^n$ arbitrarily d -close.

 **Definition. Let**

$$V_j(m) := h_{\alpha_0 j}(B(m))$$

for $1 \leq j \leq k, 1 \leq m \leq k+1$ and

$$V'_j(m) := h_{\alpha_i j}(B(m)) \subset M'$$

for $M' = M_{\alpha_i}$ for large enough i .

📌 Definition. Now let

$$U_j(m) := \bigcup_{p=1}^j V_p(m)$$

Note $U_k(1) = M$ and $U'_k(1) = M$.

📌 Definition.

$$f_j : V_j(k+1) \rightarrow V'_j(k+1) \\ h_{\alpha_i,j} \circ h_{\alpha_0,j}^{-1}$$

Suppose we construct an imbedding

$$g_j : U_j(m) \rightarrow M'$$

ϵ -close to identity by choosing M' far enough in the sequence.

Mikhail Gromov - What is a Manifold

<https://www.ihes.fr/~gromov/wp-content/uploads/2018/08/manifolds-Poincare.pdf>

#talk

[What is a Manifold? - Mikhail Gromov - YouTube](#)

1 Ideas and Definitions.

We are fascinated by knots and links. Where does this feeling of beauty and mystery come from? To get a glimpse at the answer let us move by 25 million years in time.

25×10^6 is, roughly, what separates us from orangutans: 12 million years to our common ancestor on the phylogenetic tree and then 12 million years back by another branch of the tree to the present day orangutans.

But are there topologists among orangutans?

Yes, there definitely are: many orangutans are good at "proving" the triviality of elaborate knots, e.g. they fast master the art of untying boats from their mooring when they fancy taking rides downstream in a river, much to the annoyance of people making these knots with a different purpose in mind.

A more amazing observation was made by a zoo-psychologist Anne Russon in mid 90's at Wanariset Orangutan Reintroduction Project (see p. 114 in [68]).

"... Kinei [a juvenile male orangutan], when he was in a possession of a hose, invested every second in making giant hoops, carefully inserting one end of his hose into the other and jamming it in tight. Once he'd made his hoop, he passed various parts of himself back and forth through it – an arm, his head, his feet, his whole torso – as if completely fascinated with idea of going through the hole."

Playing with hoops and knots, where there is no visible goal or any practical gain – be it an ape or a 3D-topologist – appears fully "non-intelligent" to a practically minded observer. But we, geometers, feel thrilled at seeing an animal whose space perception is so similar to ours.



1 Looking at Manifolds from Different Angles.

1.1 Ideas and Definitions.

Enchanted Knots and links and Kinei passing himself back and forth through rings.

Are we "equivalence classes of atlases" and "ringed spaces"?

Where do orangutans and zoo-psychologists disagree in defining "tools"?

1.2 Sharp Turns in Topology

Geometric/analytic structures:

Abel, Riemann, Picard, Poincaré, Klein

"naked" algebraic topology:

Poincaré-Whitney-Serre-Milnor-Smale.

Who is naked?

Why can no geometric flow round up d -spheres for $d \geq 4$?

Geometry is back:

- Poincaré's "naked" algebraic topology and little bit of geometry will tell you everything in the low dim



2 Homotopies and Obstructions.

For more than half a century, starting from Poincaré, topologists have been laboriously stripping their beloved science of its geometric garments.

"Naked topology", reinforced by homological algebra, reached its to-day breathtakingly high plateau with the following

-

N	n	homotopy classes of maps $S^{n+N} \rightarrow S^N$
any	0	$\mathbb{Z}$; index
even	$2N - 1$	contains a subgroup of finite index isomorphic to $\mathbb{Z}$

 **(Serre)** There are at most finitely many homotopy classes of maps $S^{n+N} \rightarrow S^N$ except for two cases

N	n	homotopy classes of maps $S^{n+N} \rightarrow S^N$
any	0	$\mathbb{Z}$; index
even	$2N - 1$	contains a subgroup of finite index isomorphic to $\mathbb{Z}$

- Whitney 1940 - geometry of embedded non-orientable surface in $\mathbb{R}^4$
 - Analysis to topology
 - Donaldson:

Atiyah-Singer
Whitney Conjecture
Thurston-Donaldson
Perelman: Energy versus Entropy
Why the proofs of the Poincare conjecture for $d = 3$ and $d \geq 5$ have nothing in common?

1.3 What is so special about the numbers 2,3,4?

Non-history: We learned *first* that the equations of degree $d \geq 5$ are unsolvable and only later mysterious formulas were found for $d = 3,4$ to everybody's delight and amazement.

$$\dots \sqrt{\dots + \frac{\sqrt[3]{2}(c^2-3bd+12c)}{3\sqrt[3]{2c^3+\dots-27b^2c-72cc+\sqrt{-4(c^2-3bd+12c)^3+(2c^3+\dots+27b^2c)}}}}$$

No search for a comparably "beautiful something" for all $d \geq 5$.
No Galois theory, no class field theory, no Langlands program no Wiles' theorem.
A dream of "functorial topological/geometric theory" incorporating Thurston-Donaldson-Perelman.

Hyperbolic 3-manifolds "comes" from any 3-manifold

- Never before non-linear PDEs were saying something about topology
- Thurston, Donaldson, Hamilton (https://en.wikipedia.org/wiki/Richard_S._Hamilton?)



$$4 =_3 2 + 2$$

2 What are we?

2.1 Seven Constructions of Manifolds.

Triangulations and surgeries: counting and limitations.

Lie groups and locally homogeneous spaces.

Algebraic Equations.

Genericity and Transversality: is randomness at the heart of manifolds?

Markov partitions and Markov spaces.

Singular and non-singular solutions of elliptic variational problems.

Quote

Proof of Donaldson theorem is $4 = 2 + 2$.

Riemannian connection

- *Torsion-free*

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

- *metric-compatible*

$$X(g(Y, Z)) = g(\nabla_X Z) + g(Y, \nabla_X Z)$$

$$\begin{aligned} \nabla : \text{Vec}(M) &\rightarrow \text{Vec}(M) \otimes \Omega^1(M) = ? \Gamma(T^*M \otimes TM) \\ X &\mapsto \nabla_{(-)} X \end{aligned}$$

$$[X, Y] \mapsto \nabla[X, Y] = ?$$

expression

We compute

$$X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y))$$

and obtain

$$g(\nabla_X Y, Z) = \begin{aligned} &X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &- g(Y, [X, Z]) - g(Z, [Y, X]) + g(X, [Z, Y]) \end{aligned}$$

the covector field of $\nabla_X Y$ is

$$X(g(Y, -)) + Y(g(-, X)) - d(g(X, Y)) - g(Y, [X, -]) - g(-, [Y, X]) + g(X, [-, Y])$$

in a chart

On a chart (U, x) we have

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_m \Gamma_{ij}^m \frac{\partial}{\partial x_m}$$

because $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = 0$ we have

Parallel transport on a curve in a semi-Riemannian manifold

Definition. Parallel vector or tensor fields along curves

Let (M, ∇) be a smooth manifold with a connection on its tangent bundle. A smooth vector or tensor field

$$V \in \mathbf{T}^{r,s}T_\gamma M$$

along a smooth curve

$$\gamma : I \rightarrow M$$

is said to be **parallel along γ** (with respect to ∇) if

$$\nabla_{\dot{\gamma}} V \equiv 0$$

Locally, along a curve

$$x(\gamma(t)) = (\gamma_1(t), \dots, \gamma_n(t))$$

and vector field

$$V(t) = \sum_{i=1}^n V_i(t) \frac{\partial}{\partial x_i}$$

is *parallel* $\iff$

$$(\nabla_{\dot{\gamma}} V)_k = \dot{V}_k + \sum_{i,j} \Gamma_{ij}^k V_j \dot{\gamma}_i = 0$$

for all components $1 \leq k \leq n$.

Curvature of a Riemannian manifold

📌 Definition. Riemann curvature tensor of a Riemannian manifold

The assignment

$$(X,Y,Z) \mapsto ([\nabla_X,\nabla_Y] - \nabla_{[X,Y]})Z$$

produces a (1,3)-tensor on M .

The **Riemann curvature tensor** is the (0,4)-tensor

$$R(X,Y,Z,W) := g(([\nabla_X,\nabla_Y] - \nabla_{[X,Y]})Z,W)$$

☰ The Riemann curvature (0,4)-tensor is actually

$$R \in \Gamma Sym^2 \Lambda^2 TM$$

and it satisfies the two **Branchi identities**

$$R(W,X,Y,Z) + R(X,Y,W,Z) + R(Y,W,X,Z) = 0$$

📌 Definition. Ricci and scalar tensors

$$\text{Ric}(X,Y) := \text{tr}(Z \mapsto$$

		R^l_{ijk}
Riemann $Sym^2 \Lambda^2 TM$ -tensor		$R_{ijkl} = g_{lm} R^m_{ijk}$
Ricci		$R_{ij} = R^k_{kij} = g^{km} R_{imj}$
scalar	$sc := \text{tr}(\text{Ric})$	$\kappa = R^i_i = g^{ij} R_{ij}$
total scalar curvature	$\int_M sc \lambda_g$	
sectional : a section of $\mathbf{Gr}^2 \Lambda^2 TM$ or a bilinear form on $\Lambda^2 TM$	$\kappa(v,w) := \frac{R(v,w,w,v)}{ v \wedge w ^2}$	
section of $T^u M$	$\kappa(v,v)$	

List of Riemannian manifolds

Riemannian manifold	sectional curvature	Ricci	scalar
$\mathbb{R}^n$ with standard metric	0		
S^n of radius R	R^{-2}		
H^n of "radius" R	$-R^{-2}$		
$S^n \times \mathbb{R}^m$			

integration by parts and pieces

vector calculus in $\mathbb{R}^3$

space.R.3.Vec.derivatives

Derivatives of vector fields in $(\mathbb{R}^3, \text{DOT})$

$$\begin{array}{ccccc} \mathcal{C}^\infty(U) & \xrightarrow{\text{grad}} & \text{Vec}^\infty(U) & \xrightarrow{\text{curl}} & \text{Vec}^\infty(U) & \xrightarrow{\text{div}} & \mathcal{C}^\infty(U) \\ & & \text{curl} \circ \text{grad} = 0 & & \text{div} \circ \text{curl} = 0 & & \end{array}$$

- and we have directional derivative of scalar fields

$$\begin{aligned} \partial_X : \mathcal{C}^\infty(U) &\rightarrow \mathcal{C}^\infty(U) \\ \partial_X \phi &= \langle X, \text{grad}(\phi) \rangle = d\phi(X) \end{aligned}$$

- and vector fields

$$\partial_X : \text{Vec}^\infty(U) \rightarrow \text{Vec}^\infty(U)$$

- and the Laplacian of scalar fields

$$\Delta : \mathcal{C}^\infty(U) \rightarrow \mathcal{C}^\infty(U)$$

- and vector fields

$$\Delta : \text{Vec}^\infty(U) \rightarrow \text{Vec}^\infty(U)$$

Note, curl, directional derivatives and two Laplacians are $\mathbb{R}$ -vector space *endomorphisms*, thus we may talk about invariant subspaces, etc.

- The Leibnitz-type product rules which all comes from

$$d^{k+1}\omega \wedge \eta = (d^k\omega) \wedge \eta + (-1)^k \omega \wedge d^l\eta$$

giving us

$$\begin{aligned} \partial_X(fg) &= f\partial_X g + g\partial_X f \\ \text{grad}(fg) &= f\text{grad}(g) + g\text{grad}(f) \\ \text{curl}(fX) &= f\text{curl}(A) - A \times \text{grad}(f) \\ \text{div}(fX) &= f\text{div}(A) + A \cdot \text{grad}(f) \end{aligned}$$

- On

$$(\text{Vec}^\infty(U), +, \times, \langle, \rangle)$$

where

$$\langle, \rangle : \text{Vec}^\infty(U) \times \text{Vec}^\infty(U) \rightarrow \mathcal{C}^\infty(U)$$

we have more "products", so more "product" rules

$$\begin{aligned} \text{grad} \langle X, Y \rangle &= X \times \text{curl}(Y) + Y \times \text{curl}(X) + \partial_X Y + \partial_Y X \\ \text{curl}(X \times Y) &= -\partial_X Y + \partial_Y X + X \text{div}(Y) - Y \text{div}(X) \\ \text{div}(X \times Y) &= \langle Y, \text{curl}(X) \rangle - \langle X, \text{curl}(Y) \rangle \end{aligned}$$

- And this one connects double derivatives

$$\operatorname{curl} \circ \operatorname{curl} = \operatorname{grad} \circ \operatorname{div} - \Delta$$

$X = X^i \partial_{x_i}$	$\alpha = X^i \mathrm{d}x_i$
$\operatorname{curl}(X^i \partial_{x_i})$	$\mathrm{d}(X^i \mathrm{d}x_i) = \frac{\partial X^i}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_i$
	$\iota_X \mathrm{d}(X^i \mathrm{d}x_i) = \frac{\partial X^i}{\partial x_j} \mathrm{d}x_j \wedge \mathrm{d}x_i(X, -)$
	$\alpha \wedge \beta$

calculus in $\mathbb{R}^n$

object	In Cartesian coordinates
Volume form λ	$\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$

integrating *gradient vector field* $\otimes$ *volume form* in $\mathbb{R}^n$

- Let $B \subset \mathbb{R}^n$ be a bounded open set with $\mathcal{C}^1$ boundary. Let $u \in \mathcal{C}^1(\text{open nbd of } \bar{\Omega})$ then we have

$$^{n-1}\alpha = u \widehat{\mathrm{d}x_i}$$

where $\widehat{\mathrm{d}x_i}$ is $\lambda = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ with $\mathrm{d}x_i$ removed.

- The **exterior derivative** of this $n - 1$ form is the n -form

$$\mathrm{d}\alpha = \frac{\partial u}{\partial x_i} \mathrm{d}x_i \wedge \widehat{\mathrm{d}x_i} = (-1)^{i+1} \frac{\partial u}{\partial x_i} \lambda$$

where

$$\begin{aligned} \mathrm{d}x_i \wedge \widehat{\mathrm{d}x_i} &= \mathrm{d}x_i \wedge \mathrm{d}x_1 \wedge \cdots \mathrm{d}x_{i-1} \wedge \mathrm{d}x_{i+1} \wedge \cdots \wedge \mathrm{d}x_n \\ &= (-1)^{i+1} \underbrace{\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n}_{\lambda} \end{aligned}$$

- Then by *Stroke's theorem*

$$\begin{aligned} \int_B \mathrm{d}\alpha &= (-1)^{i+1} \int_{\partial B} \alpha \\ \implies \int_B \frac{\partial u}{\partial x_i} \lambda &= (-1)^{i+1} \int_{\partial B} u \widehat{\mathrm{d}x_i} \end{aligned}$$

Example

For $n = 2$ we have

$$\int_{\partial B} -u \mathrm{d}x = \int_B \frac{\partial u}{\partial y} \mathrm{d}x \wedge \mathrm{d}y$$

- Let us parameterize ∂B by a $\mathcal{C}^1$ function

$$\begin{aligned} U(\text{open}) &\subseteq \mathbb{R}^{n-1} \rightarrow B \\ y &\mapsto (g(y), y) \end{aligned}$$

where $g \in \mathcal{C}^1(U, \mathbb{R})$.

- Then

$$\begin{aligned} \int_{\partial B} u \widehat{dx}_i &= \int_U u(g(y), y) dg \wedge \widehat{dy}_{i-1} \\ &= \int_U u(g(y), y) \frac{\partial g}{\partial y_{i-1}} dy_{i-1} \wedge \widehat{dy}_{i-1} \\ &= (-1)^i \int_U u(g(y), y) \frac{\partial g}{\partial y_{i-1}} dy_1 \wedge \cdots \wedge dy_{n-1} \end{aligned}$$

for $i > 1$ and

$$= \int_U u(g(y), y) dy_1 \wedge \cdots \wedge dy_{n-1}$$

for $i = 1$.

- Thus

$$\int_B \frac{\partial u}{\partial x_i} \lambda = - \int_U u(g(y), y) \frac{\partial g}{\partial y_{i-1}} dy_1 \wedge \cdots \wedge dy_{n-1}$$

for $i < n$ and

$$= \int_U u(g(y), y) dy_1 \wedge \cdots \wedge dy_{n-1}$$

for $i = 1$.

- Tensoring with $\hat{e}_i$ and summing for $1 \leq i \leq n$ we have

$$\int_B \text{grad}(u) \otimes \lambda = \int_U u(1, -\text{grad}(g)) \otimes (dy_1 \wedge \cdots \wedge dy_{n-1})$$

where $X \otimes \lambda$ is a vector field $\otimes n$ -form, which we understand is integrated component-wise.

- We identify

$$(1, -\text{grad}(g)) \otimes (dy_1 \wedge \cdots \wedge dy_{n-1}) =: \hat{N}_{\partial B} \otimes \lambda_{\partial B}$$

then we have

$$\int_B \text{grad}(u) \otimes \lambda_B = \int_{\partial B} u \hat{N}_{\partial B} \otimes \lambda_{\partial B}$$

integration by parts in $\mathbb{R}^n$

calculus on a Riemannian manifold

divergence theorem

Let (M, g) be a **compact** manifold with boundary and $X \in \text{Vec}(M)$. We know

$$d(\iota_X \lambda_g) = \mathcal{L}_X \lambda_g = (\text{div}_g(X)) \lambda_g$$

and

$$\lambda_{\hat{g}} = \iota_{\hat{N}} \lambda_g$$

where $\hat{N}$ is the outward unit normal to ∂M and $\hat{g}$ is the induced metric on ∂M .

Then

$$\begin{aligned} X &= g(X, \hat{N}) \hat{N} + X^\perp \\ \implies \iota_X \lambda_g &= \lambda_g(g(X, \hat{N}) \hat{N} + X^\perp, -) \\ &= g(X, \hat{N}) \lambda_{\hat{g}} + 0 \\ \iota_X \lambda_g &= g(X, \hat{N}) \lambda_{\hat{g}} \end{aligned}$$

Then for M **oriented**, by *Stokes theorem* we have

$$\int_M \operatorname{div}_g(X) \lambda_g = \int_{\partial M} \langle X, N \rangle_g \lambda_{\hat{g}}$$

When M is not orientable we have

$$\int_M \operatorname{div}(X) \mu_g = \int_{\partial M} g(X, N) \mu_{\hat{g}}$$

integration by parts

Because

$$\operatorname{div}(uX) = u \operatorname{div}(X) + Xu$$

we have

$$\int_M Xu \lambda_g = \int_{\partial M}$$

$$L^2(D^2)$$

For every L^2 function on the unit disk

$$u : D^2 \rightarrow \mathbb{C}$$

we can expand

$$D^2 \setminus \{0\} \cong (0, 1) \times \frac{\mathbb{R}}{\mathbb{Z}} \rightarrow \mathbb{C}$$

$$u(r, x) = \sum_{n \geq 0} a_n(r) e^{2\pi i n x}$$

The Green's function for Laplacian on unit disk

$$G(z, w) = \frac{1}{2\pi} \log \left| \frac{z - w}{w(z - \bar{w})} \right|$$

 The Laplacian is unbounded. But we have the compact inverse *integral* operator which solves the Poisson equation?

The Fourier-Bessel functions

$$\{J_n(\lambda_{n,m}r) \cos(n\theta), J_n(\lambda_{n,m}r) \sin(n\theta)\}$$

form an orthogonal basis for $L^2(D^2)$. Moreover they are eigenfunctions for the Laplacian on the standard metric on D^2 .

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.special import jn_zeros

# Parameters
R = 1
m_max = 5 # Maximum m value to consider
n_max = 5 # Maximum n value to consider

# Compute eigenvalues
eigenvalues = []
for m in range(m_max + 1):
    zeros = jn_zeros(m, n_max)
    eigenvalues.extend(zeros**2 / R**2)

# Sort eigenvalues
eigenvalues = np.sort(eigenvalues)
print(eigenvalues)

# Plot the spectrum on R
plt.figure(figsize=(9, 5))
plt.scatter(eigenvalues, np.zeros_like(eigenvalues), marker='|', color='b', s=100)
plt.xlabel("Eigenvalues of the Laplacian on the unit disk")

plt.title("Spectrum of the Laplacian on the Unit Disk (Dirichlet BCs)")
plt.grid(True, axis='x', linestyle='--', alpha=0.5)
```

```
plt.show()
```

```
from scipy import special
import numpy as np
def drumhead_height(n, k, distance, angle, t):
    kth_zero = special.jn_zeros(n, k)[-1]
    return np.cos(t) * np.cos(n*angle) * special.jn(n, distance*kth_zero)

theta = np.r_[0:2*np.pi:50j]
radius = np.r_[0:1:50j]
x = np.array([r * np.cos(theta) for r in radius])
y = np.array([r * np.sin(theta) for r in radius])
z = np.array([drumhead_height(1, 1, r, theta, 0.5) for r in radius])

import matplotlib.pyplot as plt
fig = plt.figure()
ax = fig.add_axes(rect=(0, 0.05, 0.95, 0.95), projection='3d')
ax.plot_surface(x, y, z, rstride=1, cstride=1, cmap='RdBu_r', vmin=-0.5, vmax=0.5)
ax.set_xlabel('X')
ax.set_ylabel('Y')
ax.set_xticks(np.arange(-1, 1.1, 0.5))
ax.set_yticks(np.arange(-1, 1.1, 0.5))
ax.set_zlabel('Z')
plt.show()
```

$$L^2(\Omega)$$

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set in $\mathbb{R}^n$.

Theorem 9.8 (Unique solution of the elliptic boundary value problem). *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let the operator L in (9.20) be uniformly elliptic, with coefficients $a^{ij} \in \mathbf{L}^\infty(\Omega)$. Then, for every $f \in \mathbf{L}^2(\Omega)$, the boundary value problem (9.19) has a unique weak solution $u \in H_0^1(\Omega)$. The corresponding solution operator, which we denote as $L^{-1} : f \mapsto u$, is a compact linear operator from $\mathbf{L}^2(\Omega)$ into $H_0^1(\Omega)$.*

Theorem 9.9 (Representation of solutions as a series of eigenfunctions). Assume $a^{ij} = a^{ji} \in \mathbf{L}^\infty(\Omega)$. Then, in the setting of Theorem 9.8, the linear operator $L^{-1} : \mathbf{L}^2(\Omega) \mapsto \mathbf{L}^2(\Omega)$ is compact, one-to-one, and self-adjoint.

The space $\mathbf{L}^2(\Omega)$ admits a countable orthonormal basis $\{\phi_k; k \geq 1\}$ consisting of eigenfunctions of L^{-1} , and one has the representation

$$(9.26) \qquad L^{-1}f \; = \; \sum_{k=1}^\infty \lambda_k (f, \phi_k)_{\mathbf{L}^2} \phi_k \; .$$

The corresponding eigenvalues λ_k satisfy

$$(9.27) \qquad \lim_{k \rightarrow \infty} \lambda_k = 0 \, , \qquad \lambda_k > 0 \text{ for all } k \geq 1.$$

[1]

$$L^2(M,g)$$

[2]

[3]

1. <https://math.stackexchange.com/a/1707821/1290493> ↩
2. <https://math.stackexchange.com/q/1754700/1290493> ↩
3. <https://www.math.mcgill.ca/toth/spectral%20geometry.pdf> ↩

IMPA (2019) Functional analysis

[Doctorate program: Functional Analysis \(2019\) - YouTube](#)

Sobolev spaces and generalized derivatives on $\mathbb{R}^N$

Let

$$G \subset \mathbb{R}^N$$

be an open, bounded set.

Consider the *test* functions

$$\mathcal{C}_c^\infty(G)$$

which are smooth functions on G with **compact support**, whose value on ∂G is defined to be 0.

We have by integration by parts that

$$\int_G (\partial_i u) v = - \int_G u (\partial_i v)$$

for $u, v \in \mathcal{C}_c^\infty(G)$ because the boundary values of u, v is 0.

Then

Definition. Generalized derivatives of $L^2(G)$

Let $G \subset \mathbb{R}^N$ be an open, bounded set. If for $u, w \in L^2(G)$ we have

$$f \in \mathcal{C}_c^\infty(G) \implies \int_G w f = - \int_G u (\partial_j f)$$

Then we say the **generalized derivative** of u in the direction $\hat{x}_j$ is w that is

$$\partial_j u = w$$

on G .

We observe

- If there are *two* $w_1, w_2 \in L^2(G)$ such that

$$f \in \mathcal{C}_c^\infty(G) \implies \int_G w_1 f = - \int_G u (\partial_j f) = \int_G w_2 f$$

then

$$\int_G (w_1 - w_2) f = 0$$

for all compactly supported smooth functions f on G .

- As $\mathcal{C}_c^\infty(G)$ is dense in $L^2(G)$ we have a sequence $f_n \in \mathcal{C}_c^\infty(G)$

$$f_n \xrightarrow{n \rightarrow \infty, L^2(G)} f$$

we have

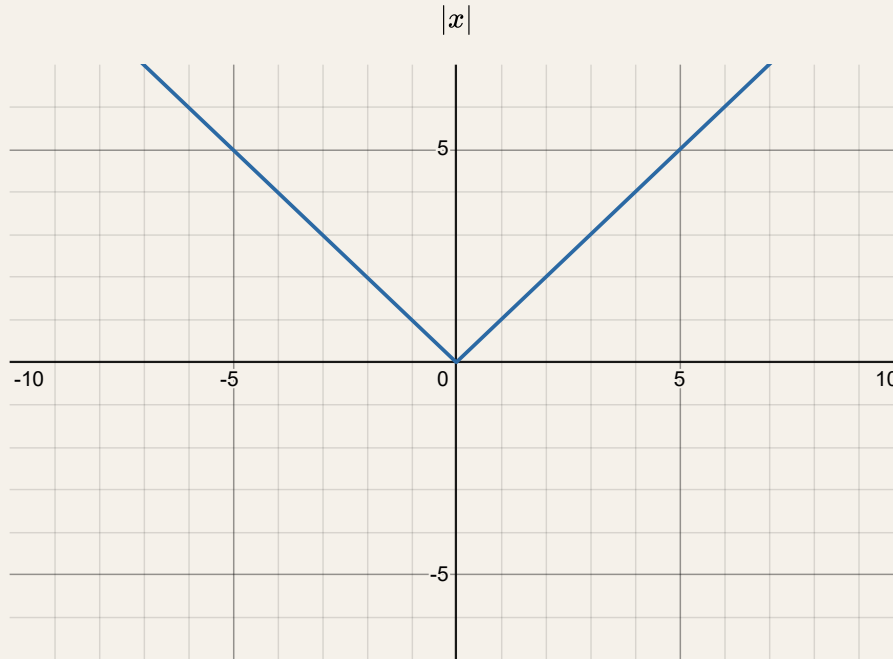
$$\int_G (w_1 - w_2) f_n = 0 \implies \int_G (w_1 - w_2) f = 0$$

Thus as f is arbitrary, $w_1 = w_2$ almost everywhere.

- Thus, generalized derivatives are unique almost everywhere, that is, unique in $L^2(G)$.

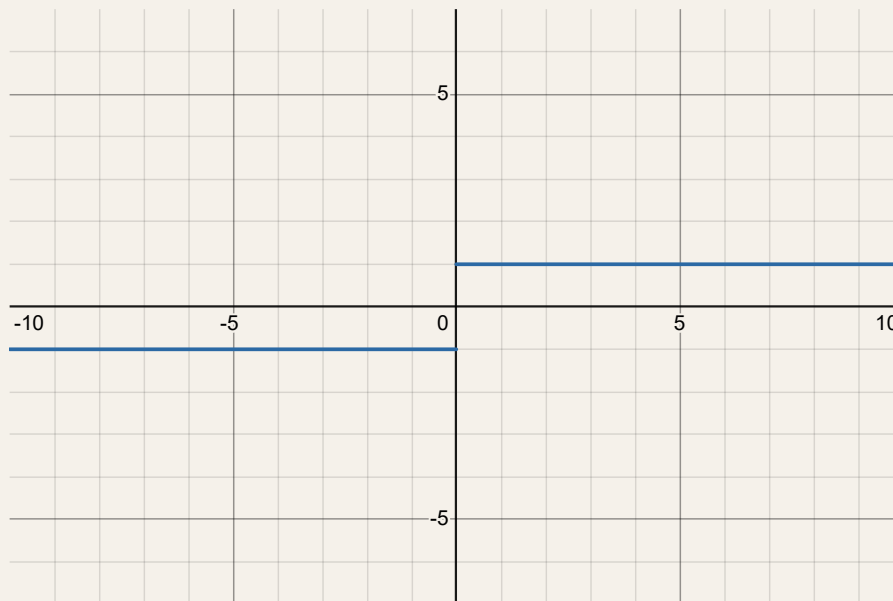
Example

For $G = (-1, 1)$ the function



The *generalized derivative* of this function is

$$\begin{cases} -1 & x < 0 \\ \text{any } c \in \mathbb{R} & x = 0 \\ 1 & x > 0 \end{cases}$$

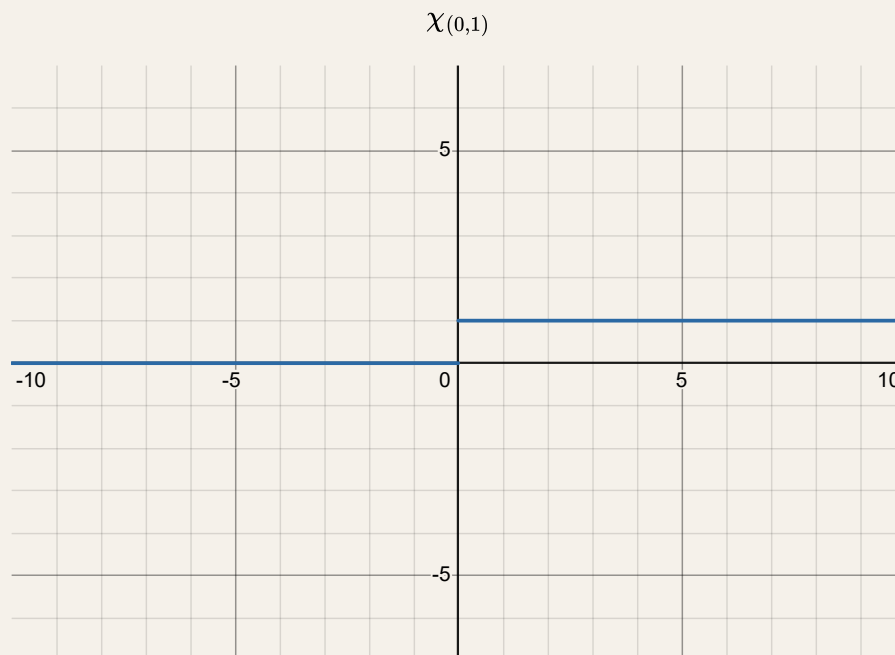


as

$$-\int_{(-1,1)} |x| f(x) = -\int_{(-1,0)} (-1) f(x) - \int_{(0,1)} (1) f(x)$$

Non-existence of generalized derivative in $L^2(G)$

For $G = (-1, 1)$, the *generalized derivative* of



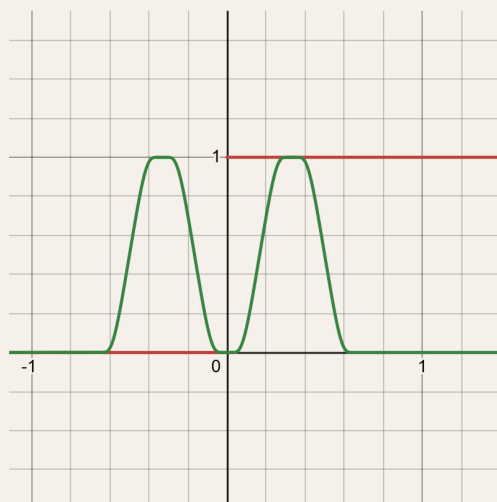
w must satisfy

$$-\int_{(-1,1)} wf = \int_{(-1,1)} \chi_{(0,1)} f' = \int_{(0,1)} f' = -f(0)$$

for all $f \in \mathcal{C}_c^\infty(G)$.

Then for every such f with support outside $(-\epsilon, \epsilon)$ we have

$$\int_G wf = 0$$



thus $w = 0$ on $(-\epsilon, \epsilon)$ for all $\epsilon > 0$. Then w is only non-zero at $0 \in (-1, 1)$, but this means

$$\int_{(-1,1)} wf = \int_{(-1,1) \setminus \{0\}} w^0 f = 0$$

Hence, $\chi_{(0,1)}$ does not have a generalized derivative in $L^2(G)$.

derivative of distributions

What we must do is define the derivative of $\chi_{(0,1)}$ as the *linear functional*

$$\frac{d}{dx} \chi_{(0,1)} = \delta_0 : \mathcal{C}_c^\infty(-1,1) \rightarrow \mathbb{R} \\ f \mapsto f(0)$$

Example

The derivative of the Dirac delta

$$\delta_0 : \mathcal{C}_c^\infty(-1,1) \rightarrow \mathbb{R} \\ f \mapsto f(0)$$

is the distribution

$$\mathcal{C}_c^\infty(-1,1) \rightarrow \mathbb{R} \\ f \mapsto -f'(0)$$

denoted by δ'_0 . Hence, we have the *generalized derivatives*

$$\dots \xrightarrow{\frac{d}{dx}} \frac{x^k}{k!} \chi_{(0,1)} \dots \xrightarrow{\frac{d}{dx}} \frac{x^2}{2} \chi_{(0,1)} \xrightarrow{\frac{d}{dx}} x \chi_{(0,1)}(x) \xrightarrow{\frac{d}{dx}} \chi_{(0,1)} \xrightarrow{\frac{d}{dx}} \delta'_0 \xrightarrow{\frac{d}{dx}} \delta''_0 \xrightarrow{\frac{d}{dx}} \dots \xrightarrow{\frac{d}{dx}} \delta_0^{(k)} \xrightarrow{\frac{d}{dx}} \dots$$

derived by integrating "functions" starting from the right, where

$$\delta_0^{(k)} : \mathcal{C}_c^\infty(-1,1) \rightarrow \mathbb{R} \\ f \mapsto (-1)^k f^{(k)}(0)$$

Intuition

A intuition behind the derivative of Delta:

* To find $\langle x | \hat{p} \rangle$

(input) $\hat{x} \hat{p} - \hat{p} \hat{x} = i\hbar \mathbb{1}$ (defn of canonically conj variables)

$$\langle x | \hat{x} \hat{p} - \hat{p} \hat{x} | x' \rangle = i\hbar \langle x | \mathbb{1} | x' \rangle$$

$$\langle x | x' \rangle = \delta(x-x')$$

$$x \langle x | \hat{p} | x' \rangle - x' \langle x | \hat{p} | x' \rangle = i\hbar \delta(x-x')$$

$$\langle x | \hat{p} | x' \rangle = \frac{i\hbar \delta(x-x')}{x-x'}$$

(non-rigorous) $\rightarrow \int \delta(x) \rightarrow \int \delta'(x) = -\frac{\delta(x)}{x}$

$$\therefore = i\hbar \frac{\partial}{\partial x} \delta(x-x')$$

$$\langle x | \hat{p} | x' \rangle = -i\hbar \frac{\partial}{\partial x} \langle x | x' \rangle$$

$$\Rightarrow \boxed{\langle x | \hat{p} | \Psi \rangle = -i\hbar \frac{\partial}{\partial x} \langle x | \Psi \rangle}$$

IN POSITION BASIS

Note: $\hat{x} | x \rangle = x | x \rangle$

Definition.

$$W_2^1$$

Let $G \subset \mathbb{R}^N$ be an open, bounded set. We have the space of all $L^2(G)$ functions whose generalized derivatives exist in all directions:

$$W_2^1(G) := \left\{ u \in L^2(G) \mid \begin{array}{l} \forall 1 \leq j \leq N \exists w_j \in L^2(G) : \\ f \in \mathcal{C}_c^\infty(G) : \int_G (\partial_j f) u = - \int_G w_j f \end{array} \right\}$$

with an inner product

$$\begin{aligned} \langle u, v \rangle_{1,2} &:= \langle u, v \rangle_{L^2} + \sum_{j=1}^N \langle \partial_j u, \partial_j v \rangle_{L^2} \\ &=: \langle u, v \rangle_{L^2} + \langle \text{grad}(u), \text{grad}(v) \rangle_{L^2} \end{aligned}$$

Proposition:

- Convergence in $\langle, \rangle_{1,2}$ implies convergence in L^2
- Thus

$$(W_2^1, \langle, \rangle_{1,2})$$

is a Hilbert space.

Definition.

$$\mathring{W}_2^1(G) := \overline{(\mathcal{C}_c^\infty(G), \langle, \rangle_{1,2})}$$

Intuition

We have the *technically wrong* idea

$$\mathring{W}_2^1(G) = \{u \in W_2^1(G) \mid u|_{\partial G} = 0\}$$

A more precise statement, in dim 1, is the following:

Lemma: (For $\dim G = 1$) Let $(a, b) \subset \mathbb{R}$ be a finite interval. For every $u \in \mathring{W}_2^1(a, b)$,

$$\exists v \in \mathcal{C}^0[a, b] : u = v \text{ ae}, v(a) = 0 = v(b)$$

- Fix $x \in (a, b)$. Consider $f \in \mathcal{C}_c^\infty(a, b)$ then

$$\begin{aligned} |f(x)|^2 &= |f(x) - f(a)|^2 = \left| \int_{[a,x]} f' \right|^2 < (x-a) \int_{[a,x]} |f'|^2 \\ &\leq (b-a) \|f'\|_2^2 \end{aligned}$$

- Then

$$f \in \mathcal{C}_c^\infty(a, b) \implies \|f\|_\infty \leq \sqrt{(b-a)} \underbrace{\|f'\|_2}_{\|f\|_{1,2}}$$

☀ Let u be in

📌 Definition.

$$\mathring{W}_2^1(G) := \overline{(\mathcal{C}_c^\infty(G), \langle \cdot, \cdot \rangle_{1,2})}$$

then

$$f_n \xrightarrow{n \rightarrow \infty, \langle \cdot, \cdot \rangle_{1,2}} u$$

where $f_n \in \mathcal{C}_c^\infty(a, b)$.

- Then

$$\|f_n - f_m\|_\infty \leq \sqrt{b-a} \|f_n - f_m\|_{1,2}$$

thus f_n is Cauchy in $\|\cdot\|_\infty$. So it converges in $\mathcal{C}^0[a, b]$ to a continuous function

$$f_n \xrightarrow{n \rightarrow \infty, \|\cdot\|_\infty} f$$

- By

Proposition:

- Convergence in $\langle \cdot, \cdot \rangle_{1,2}$ implies convergence in L^2
- Thus

$$(W_2^1, \langle \cdot, \cdot \rangle_{1,2})$$

is a Hilbert space.

we know

$$f_n \xrightarrow{n \rightarrow \infty} u \in L^2(a, b)$$

- As convergence in $\|\cdot\|_\infty$ implies convergence in L^2 we know

$$u = f \text{ ae}$$

where $f \in \mathcal{C}^0[a, b]$ and

$$f(0) = \lim_{n \rightarrow \infty} f_n(0) = 0 = \lim_{n \rightarrow \infty} f_n(1) = f(1)$$

☰ Let $G \subset \mathbb{R}^N$ be an open, bounded set such that ∂G is a smooth submanifold of $\mathbb{R}^N$ (or locally smooth?), and $u \in \mathring{W}_2^1(G)$. Then

$$\int_{\partial G} u^2 \leq c \|u\|_{1,2}^2$$

☀ (Proof for G being a rectangle in $\mathbb{R}^2$)

Evans PDEs

source: [Lawrence C. Evans - Partial Differential Equations-AMS\(2010\).pdf](#)

Chapter 2

Heat equation

📌 Definition. Heat equation on $U \times (0, T)$

$$u_t - \nabla u = 0 \text{ on } U \times (0, T)$$

📌 Definition. Fundamental solution to the heat equation

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\} &\rightarrow \mathbb{R} \\ (x, t) &\mapsto G(x, t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \end{cases} \end{aligned}$$

☰ For

$$g \in \mathcal{C}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$$

and

📌 Definition. Fundamental solution to the heat equation

$$\begin{aligned} \mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\} &\rightarrow \mathbb{R} \\ (x, t) &\mapsto G(x, t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \end{cases} \end{aligned}$$

the convolution

$$u(x, t) := \int_{y \in \mathbb{R}^n} G(x - y, t) g(y)$$

is smooth $u \in \mathcal{C}^\infty(\mathbb{R}^n \times (0, \infty))$ and solves

📌 Definition. Heat equation on $U \times (0, T)$

$$u_t - \nabla u = 0 \text{ on } U \times (0, T)$$

for $T = \infty$ and for all $a \in \mathbb{R}^n$

$$\lim_{(x, t) \in \mathbb{R}^n \times (0, \infty) \rightarrow (a, 0)} = g(a)$$

📌 Definition. Heat ball

For

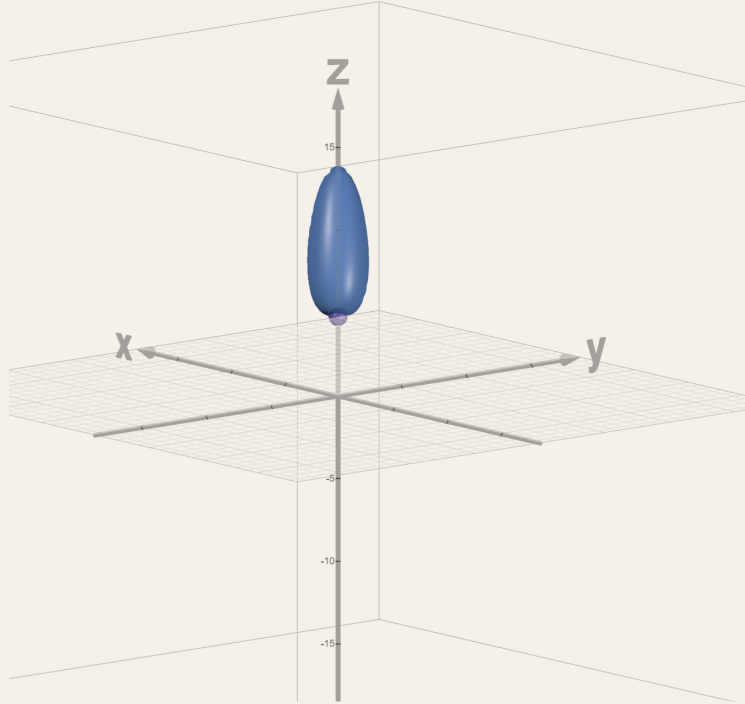
📌 Definition. Fundamental solution to the heat equation

$$\mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\} \rightarrow \mathbb{R}$$

$$(x, t) \mapsto G(x, t) := \left\{ \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \right.$$

we define the heat ball *starting* at $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ of *size* $r > 0$ as

$$E(x, t, r) := \left\{ (a, s) \in \mathbb{R}^n \times \mathbb{R} \mid s \leq t, G(a - y, t - s) \geq \frac{1}{r^n} \right\}$$



[1]

🔗 Definition. Parabolic cylinder and boundary of bounded open sets in $U \subseteq \mathbb{R}^n$

Assume $U \subseteq \mathbb{R}^n$ is **open, bounded**, and fix $T > 0$.

- the **parabolic cylinder** is

$$U_T := U \times (0, T]$$

- the **parabolic boundary** is

$$\Gamma_T := \overline{U_T} \setminus U_T = \underbrace{U \times \{0\}}_{\text{bottom}} \cup \underbrace{\partial U \times [0, T]}_{\text{sides}}$$

Hence

$$\partial U = \Gamma_T \cup \underbrace{U \times \{T\}}_{\text{top}}$$

🔗 Definition. Definition

(viii) *Functions of x and t .* It is occasionally useful to introduce spaces of functions with differing smoothness in the x - and t -variables, although there is no standard notation for such spaces. We will for this book write

$$C_1^2(U_T) = \{u : U_T \rightarrow \mathbb{R} \mid u, D_x u, D_x^2 u, u_t \in C(U_T)\}.$$

In particular, if $u \in C_1^2(U_T)$, then $u, D_x u$, etc. are continuous up to the top $U \times \{t = T\}$.

 (Mean value property for heat equation on bounded open $U \subseteq \mathbb{R}^n$) Let

$$u \in C_1^2(U_T)$$

solves the heat equation. Then

$$u(x, t) = \frac{1}{4r^n} \int_{E(x, t, r)} u(a, s) \frac{|a - y|^2}{(t - s)^2} dy \wedge ds$$

whenever $E(x, t, r) \subset U_T$.



$$\begin{aligned} \phi(r) &:= \frac{1}{r^n} \int_{E(0, 0, r)} u(a, s) \frac{|a|^2}{s^2} dy \wedge ds \\ &= \int_{E(0, 0, 1)} u(ra, r^2 s) \frac{|a|^2}{s^2} dy \wedge ds \end{aligned}$$

- Then

$$\phi'(r) = \int_{E(0, 0, 1)}$$

Chapter 3

- **First-order PDE:**

$$\begin{aligned} F(d_x u, u(x), x) &= 0 \text{ on } x \in U \\ u &= g \text{ on } \Gamma \subseteq \partial U \end{aligned}$$

- **complete integral:** $I(x; a)$ is a complete integral for F on $U \times A$ if

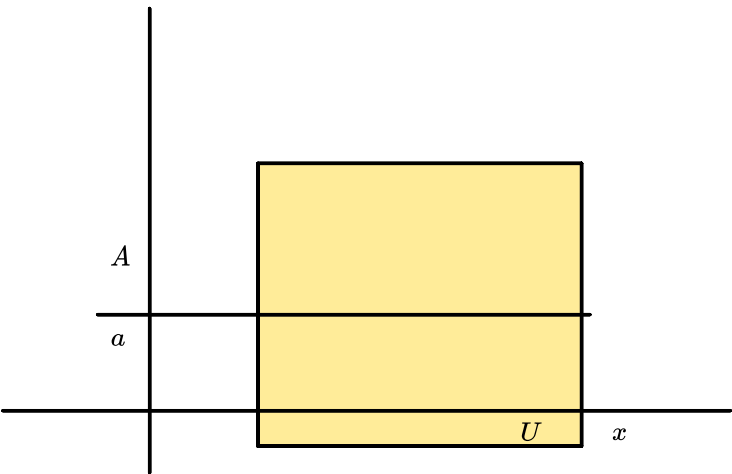
$$u(x) = I(x, a)$$

solves the PDE

- **First-order PDE:**

$$\begin{aligned} F(d_x u, u(x), x) &= 0 \text{ on } x \in U \\ u &= g \text{ on } \Gamma \subseteq \partial U \end{aligned}$$

which means



$$F(d_x I|_{U \times \{a\}}, I|_{U \times \{a\}}(x), x) = 0$$

for each $a \in A$ and

$$\text{rank} \left(\begin{pmatrix} I \\ \mathfrak{D} \begin{bmatrix} \partial_{x_1,0} I \\ \partial_{x_2,0} I \\ \vdots \\ \partial_{x_n,0} I \end{bmatrix}_{(x,-)} \end{pmatrix}^\top \right) = n$$

for each $x \in U$.

PDE	complete integral
$du(x) + f(du) = u$	$I(x; a) = a \cdot x + f(a)$ $a \in \mathbb{R}^n$
$\ du\ = 1$	$a \cdot x + b$ $a \in \mathbb{R}^n, \ a\ = 1, b \in \mathbb{R}$
$u_{x_{n+1}} + H(du) = 0$	$a \cdot x - tH(a) + b$

• Let

- **complete integral:** $I(x; a)$ is a complete integral for F on $U \times A$ if

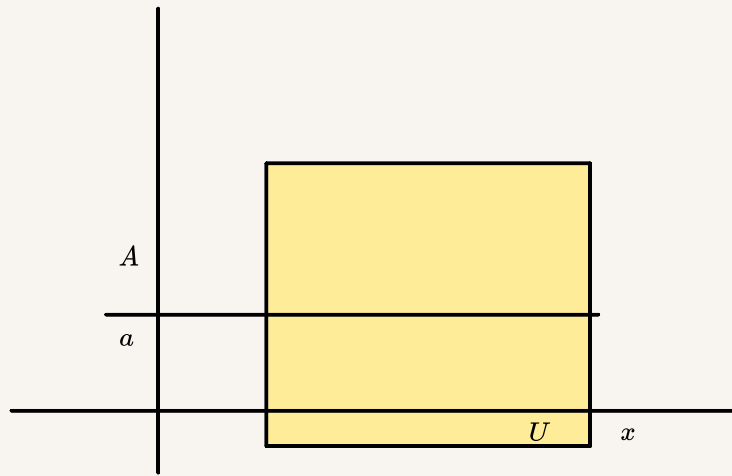
$$u(x) = I(x, a)$$

solves the PDE

- **First-order PDE:**

$$\begin{aligned} F(d_x u, u(x), x) &= 0 \text{ on } x \in U \\ u &= g \text{ on } \Gamma \subseteq \partial U \end{aligned}$$

which means



$$F(d_x I|_{U \times \{a\}}, I|_{U \times \{a\}}(x), x) = 0$$

for each $a \in A$ and

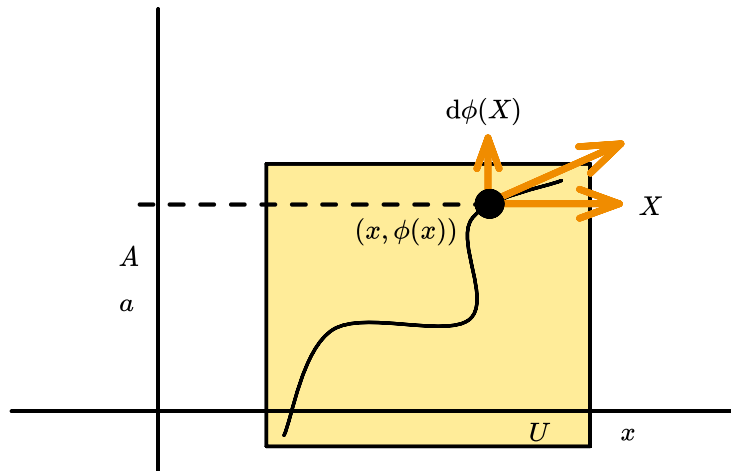
$$\text{rank} \left(\begin{pmatrix} I \\ \mathfrak{D} \begin{bmatrix} \partial_{x_1,0} I \\ \partial_{x_2,0} I \\ \vdots \\ \partial_{x_n,0} I \end{bmatrix}_{(x,-)} \end{pmatrix}^\top \right) = n$$

for each $x \in U$.

- Let $\phi : U \rightarrow A$ is a $\mathcal{C}^1$ function such that

$$d_{(x,\phi(x))} I \begin{bmatrix} 0 \\ d\phi(X) \end{bmatrix} = 0$$

- Then the function $v(x) := I(x, \phi(x))$ is called **singular integrals** and it satisfies



$$d_x v(X) = d_{(x,\phi(x))} I \begin{bmatrix} X \\ 0 \end{bmatrix} + d_{(x,\phi(x))} I \begin{bmatrix} 0 \\ d\phi(X) \end{bmatrix}$$

which means

$$d_x v = d_x I|_{U \times \{a\}}$$

hence because $I(-, a)$ is a solution of the PDE with F , v also satisfies

$$F(d_x v, v(x), x) = F(d_x I|_{U \times \{a\}}, I|_{U \times \{a\}}(x), x) = 0$$

- **general integrals** (rather than "complete integrals") are of the form

$$J(x;a') = I(x;a',h(a'))$$

PDE	complete integral, general integrals	singular integrals,
$u^2(1+\ Du\ ^2)=1$	$I(x,a)=\pm\sqrt{1-\ x-a\ ^2}$ for $\ x-a\ <1$	$\mathrm{d}_{(x,\phi(x))}I^\top=\mp\frac{x-a}{\sqrt{1-\ x-a\ ^2}}$ which is zero iff $a=\phi(x)=x$, but then $I(x,\phi(x))=\pm 1$
$u_{x_{n+1}}+H(\mathrm{d}u)=0$ for $H(p)=\ p^2\$	for $b=0$ $I(x,t;a):=a\cdot x-tH(a)$ $=a\cdot x-t\ a^2\ $	$\mathrm{d}_{(x,\phi(x))}I^\top=x-2ta$ which is zero iff $a=\phi(x,t)=\frac{x}{2t}$ $v(x)=\left(\frac{x}{2t}\right)\cdot x-t\left\ \frac{x}{2t}\right\ ^2=\frac{1}{4}\frac{\ x\ ^2}{4t}$ must solve the Hamilton-Jacobi equations

- **characteristic ODE:** Say u solves the PDE $F(\mathrm{d}u,u,x)=0$
 - and then define

$$\begin{aligned} z(s) &:= u(x(s)) \\ p(s) &:= \mathrm{d}_{x(s)}u \end{aligned}$$

then

$$\begin{aligned} z'(s) &= \mathrm{d}_{x(s)}u(x'(s)) \\ p'(s) &= \mathfrak{D}_{x(s)}^2u(-,x'(s)) \end{aligned}$$

- Because

$$F(\mathrm{d}u,u,x)=0$$

for all $x\in U$ we have

$$\left(\mathfrak{D}F(\mathfrak{D}^2u(-,X),0,0)+\mathfrak{D}F(0,\mathrm{d}u(X),0)+\mathfrak{D}F(0,0,X)\right)\Big|_{(\mathrm{d}u,u,x)}=0$$

for all $X\in V$.

- $F:V^*\times\mathbb{R}\times U\rightarrow\mathbb{R}$

so the derivative in the first slot

$$\mathfrak{D}F(-,0,0):V^*\rightarrow\mathbb{R}$$

is naturally corresponds to an element of V by $V\cong V^{**}$.

- We set

$$\mathfrak{D}F(\underbrace{\varphi}_{\in V^*},0,0)=\varphi(x')$$

- By

- and then define

$$\begin{aligned} z(s) &:= u(x(s)) \\ p(s) &:= d_{x(s)}u \end{aligned}$$

then

$$\begin{aligned} z'(s) &= d_{x(s)}u(x'(s)) \\ p'(s) &= \mathfrak{D}_{x(s)}^2 u(-, x'(s)) \end{aligned}$$

$$\mathfrak{D}F(\mathfrak{D}^2 u(-, -), 0, 0) = \dot{p} = \mathfrak{D}^2 u(-, x')$$

we get

$$\left(\cancel{\mathfrak{D}F(\mathfrak{D}^2 u(-, -), 0, 0)}^{\dot{p}(s)} + \mathfrak{D}F(0, \cancel{du(X)}^{p(s)}, 0) + \mathfrak{D}F(0, 0, -) \right) \Big|_{(s)} = 0$$

which means

$$\dot{p}_{\square} = -\mathfrak{D}F(0, p(s)_{\square}, 0) - \mathfrak{D}F(0, 0, \square)$$

-

$$z'(s) = d_{x(s)}u(x'(s)) = p(s)(x'(s)) = \mathfrak{D}F(p(s), 0, 0)$$

by

- We set

$$\mathfrak{D}F(\underbrace{\varphi}_{\in V^*}, 0, 0) = \varphi(x')$$

📌 Definition. Characteristic equations for PDEs

The following equations are called **characteristic equations**

$$\begin{aligned} \mathfrak{D}F(\varphi, 0, 0) &= \varphi(x'), \quad \forall \varphi \in V^* \\ \dot{p}_{\square} &= -\mathfrak{D}F(0, p(s)_{\square}, 0) - \mathfrak{D}F(0, 0, \square) \\ z'(s) &= \mathfrak{D}F(p(s), 0, 0) \end{aligned}$$

for the PDE

$$\begin{aligned} F : V^* \times \mathbb{R} \times U &\rightarrow \mathbb{R} \\ F(du, u, x) &= 0 \end{aligned}$$

which forms an ODE in $V^* \times \mathbb{R} \times U$ and solves for

$$u(x(s)) = z(s)$$

which is obtained from some given boundary conditions.

$$\begin{aligned} \dot{x} &= \text{grad}_{(p)}(F) \\ \dot{p} &= -\text{grad}_{(x)}(F) - \text{grad}_{(z)}(F) \\ \dot{z} &= \mathfrak{D}F(p(s), 0, 0) \end{aligned}$$

PDE	characteristic equations
linear $du(B) + cu = 0$	$\dot{x} = B(x)$ $\dot{z} = -c(x)z$

PDE	characteristic equations
quasi-linear $\mathrm{d}u(B_{x,u(x)}) + c(x,u(x)) = 0$	$\dot{x} = B(x,z)$ $\dot{z} = -c(x,z)$
conservation laws $G(\mathrm{d}u, u_t, u, x, t) = u_t + \underbrace{\mathrm{div}(F_u)}_{\mathrm{d}u(F')}$ where $F : \mathbb{R} \rightarrow U$	
Hamilton-Jacobi equations $G(\mathrm{d}u, u_t, u, x, t) = u_t + H(\mathrm{d}u, x)$	$\dot{x} = \mathrm{grad}_{(p)}(H)$ $\dot{p} = -\mathrm{grad}_{(x)}(H)$ $\dot{z} = \mathrm{d}H(p, 0, 0) - H$ first two are <i>Hamilton's equations</i>

1. <https://www.desmos.com/3d/lwzefumnnn> ↩

sett.measure.cpx

$\mathbb{C}$ -measure

Definition. $\mathbb{C}$ -measures on a measure space

Let $\mathfrak{M}$ be a σ -algebra on a set X . A $\mathbb{C}$ -measure on $\mathfrak{M}$ is a function

$$\mu : \mathfrak{M} \rightarrow \mathbb{C}$$

such that

$$\{E_i\} \text{ partitions } E \in \mathfrak{M} \implies \mu(E) = \sum_{i \geq 1} \mu(E_i)$$

- The **convergence** of $\sum_{i \geq 1} \mu(E_i)$ is now a requirement for every partition of E , unlike for positive measures where the series could converge or diverge to ∞ .
- Because $E = \bigcup_{i \geq 1} E_i$ does not change under a rearrangement of (E_i) the series $\sum_{i \geq 1} \mu(E_i)$ must actually **converge absolutely**.

dominating a complex measure with positive measures

Any positive measure $(X, \mathfrak{M}, \lambda)$ that dominates a complex measure $(X, \mathfrak{M}, \mu)$, meaning

$$E \in \mathfrak{M} \implies |\mu(E)| \leq \lambda(E)$$

must have

$$\{E_i\} \text{ partitions } E \in \mathfrak{M} \implies \lambda(E) = \sum_{i \geq 1} \lambda(E_i) \geq \sum_{i \geq 1} |\mu(E_i)|$$

So $\lambda(E)$ is at least equal to the supremum of all possible sums on the right that is

Definition. The **total variation** of a complex measure $(X, \mathfrak{M}, \mu)$ is

$$\begin{aligned} |\mu| : \mathfrak{M} &\rightarrow [0, \infty) \\ E &\mapsto |\mu|(E) := \sup_{\{E_i\} \text{ partitions } E} \sum_{i \geq 1} |\mu(E_i)| \end{aligned}$$

So for any such λ we have

$$\lambda(E) \geq |\mu|(E)$$

$$|\mu|(E) \geq |\mu(E)|$$

But actually $|\mu|$ itself is a measure!

🔑 **Definition.** The **total variation** of a complex measure $(X, \mathfrak{M}, \mu)$ is

$$|\mu| : \mathfrak{M} \rightarrow [0, \infty)$$

$$E \mapsto |\mu|(E) := \sup_{\{E_i\} \text{ partitions } E} \sum_{i \geq 1} |\mu(E_i)|$$

is a positive measure on $(X, \mathfrak{M})$ and

$$|\mu|(X) < \infty$$

So our problem of dominating μ with a positive measure does have a solution.

space of all complex measures

🔑 **Definition.** Normed space of all complex measures on $(X, \mathfrak{M})$

Let $(X, \mathfrak{M})$ be a measurable space then space of all complex measures on $\mathfrak{M}$

$$\text{Meas}(X, \mathfrak{M}, \mathbb{C})$$

is a $\mathbb{C}$ -vector space with norm

$$\|\mu\| := |\mu|(X)$$

decomposition of signed measures

For a $\mathbb{R}$ -measure μ we may decompose

$$\mu^+ := \frac{1}{2}(|\mu| + \mu)$$

$$\mu^- := \frac{1}{2}(|\mu| - \mu)$$

which are positive measure called **positive** and **negative variations** of μ respectively.

Absolute continuity

decomposition of complex measures

📖 **(Lebesgue decomposition of complex measure relative to a positive measure and Radon-Nikodym description of λ_a)** Let $(X, \mathfrak{M}, \mu)$ be a positive σ -finite measure space and λ be a complex measure on $\mathfrak{M}$. Then there is a unique pair of complex measures λ_a, λ_s on $\mathfrak{M}$ such that

$$\lambda = \lambda_a + \lambda_s$$

$$\lambda_a \ll \mu$$

$$\lambda_s \perp \mu$$

called the **Lebesgue decomposition** of λ relative to μ . If λ is positive and finite, then so are λ_a, λ_s . Moreover there is a unique $h \in L^1(X, \mathfrak{M}, \mu)$ such that

$$E \in \mathfrak{M} \implies \lambda_a(E) = \int_E h \, d\mu$$

consequences of the decomposition

 **(Polar decomposition of $\mathbb{C}$ -measures)** Let $(X, \mathfrak{M}, \mu)$ be a $\mathbb{C}$ -measure space. Then there is a measurable function

$$h : X \rightarrow U(1) \subset \mathbb{C}$$

such that

$$d\mu = h d|\mu|$$

Riesz representation

sett.Top.t.2LC

Locally compact Hausdorff spaces

 **Definition.** A Hausdorff space is **locally compact** if every point has a neighborhood whose closure is compact.

2.4 Theorem Suppose K is compact and F is closed, in a topological space X . If $F \subset K$, then F is compact.

PROOF If $\{V_\alpha\}$ is an open cover of F and $W = F^c$, then $W \cup \bigcup_\alpha V_\alpha$ covers X ; hence there is a finite collection $\{V_{\alpha_i}\}$ such that

$$K \subset W \cup V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

Then $F \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. ////

Corollary If $A \subset B$ and if B has compact closure, so does A .

2.5 Theorem Suppose X is a Hausdorff space, $K \subset X$, K is compact, and $p \in K^c$. Then there are open sets U and W such that $p \in U$, $K \subset W$, and $U \cap W = \emptyset$.

PROOF If $q \in K$, the Hausdorff separation axiom implies the existence of disjoint open sets U_q and V_q , such that $p \in U_q$ and $q \in V_q$. Since K is compact, there are points $q_1, \dots, q_n \in K$ such that

$$K \subset V_{q_1} \cup \cdots \cup V_{q_n}.$$

Our requirements are then satisfied by the sets

$$U = U_{q_1} \cap \cdots \cap U_{q_n} \quad \text{and} \quad W = V_{q_1} \cup \cdots \cup V_{q_n}. \quad \text{////}$$

Corollaries

- (a) Compact subsets of Hausdorff spaces are closed.
- (b) If F is closed and K is compact in a Hausdorff space, then $F \cap K$ is compact.

Corollary (b) follows from (a) and Theorem 2.4.

positive measures and positive functions on $\mathcal{C}_c$

A finite positive Borel measure μ on X gives a functional

$$\Lambda : \mathcal{C}_c(X) \rightarrow k$$
$$\Lambda(f) := \int_{(X,\mu)} f$$

Thus functional is *positive* in the sense that if $f(X) \subset (0,1)$ then $(\Lambda f)(X) \subset (0,1)$.

[1]

The converse of this statement is true!

 **(Riesz representation of positive functionals on $\mathcal{C}_c$)** Let X be locally compact Hausdorff space and Λ be a positive linear functional on $\mathcal{C}_c(X)$. Then there exists a σ -algebra in X which contains all Borel sets in X and there exists a **unique positive Radon measure** μ in this σ -algebra such that

$$\Lambda(f) = \int_{(X,\mu)} f$$

for every $f \in \mathcal{C}_c(X)$ and for every compact $K \subseteq X$

$$\mu(K) < \infty$$

M E Taylor - PDEs I

chapter 3

solving of constant-coefficient PDE on locally on $\mathbb{R}^n$ and globally on T^n

Let

$$P \in \mathbb{C}[X]$$

then we wish to solve for $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ or $\mathcal{D}(\mathbb{R}^n)$

$$P(D)u = f \text{ on } B_R \subset \mathbb{R}^n$$

when $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ or $\mathcal{D}(\mathbb{R}^n)$ for every $R > 0$.

- $$P(D)u = f \iff P(D + \alpha)e^{-i\langle \alpha, x \rangle} = e^{-i\langle \alpha, x \rangle} f$$

- $$P(D)u = f \text{ on } B_R \iff P(D)w = h \text{ on } [0, 1]^n$$

so we may work with the problem on T^n

- $$P(\widehat{D} + \alpha) = P(k + \alpha)$$

so inversion of $P(D + \alpha)$ boils down to a lower bound of the $P(k + \alpha)$ operator

chapter 4: Sobolev spaces

Definition. Sobolev spaces in $\mathbb{R}^n$

For $\alpha \in \mathbb{R}$

$$H^\alpha(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \langle \xi \rangle^\alpha \hat{u} \in L^2(\mathbb{R}^n)\}$$

Proposition:

$$\alpha > \frac{n}{2} \implies H^\alpha(\mathbb{R}^n) \subset \mathcal{C}_b(\mathbb{R}^n)$$

💡 Fourier inversion and Cauchy-Swartz inequality.

Corollary:

$$\alpha > \frac{n}{2} + k \implies H^\alpha(\mathbb{R}^n) \subset \mathcal{C}^k(\mathbb{R}^n)$$

chapter 5: linear elliptic equations

Let

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

be a differential operator with principal symbol

$$P_m(x, \xi) = \sum_{|\alpha|=m} a_\alpha(x) \xi^\alpha$$

is invertible for $\xi \in \mathbb{R}^n \setminus \{0\}$.

Theorem 11.1. *If $P(x, D)$ is elliptic of order m and $u \in \mathcal{D}'(M)$, $P(x, D)u = f \in H^s(M)$, then $u \in H_{loc}^{s+m}(M)$, and, for each $U \subset\subset V \subset\subset M$, $\sigma < s+m$, there is an estimate*

$$(11.3) \quad \|u\|_{H^{s+m}(U)} \leq C\|P(x, D)u\|_{H^s(V)} + C\|u\|_{H^\sigma(V)}.$$

Larry Guth - MIT Differential Analysis 2

"An analyst is a person
who knows how to use
Hölder's inequality."

I heard that J. Bourgain
said this.

Loomis-Whitney volume inequality

- Consider

$$\mathbb{R} \times D^2$$

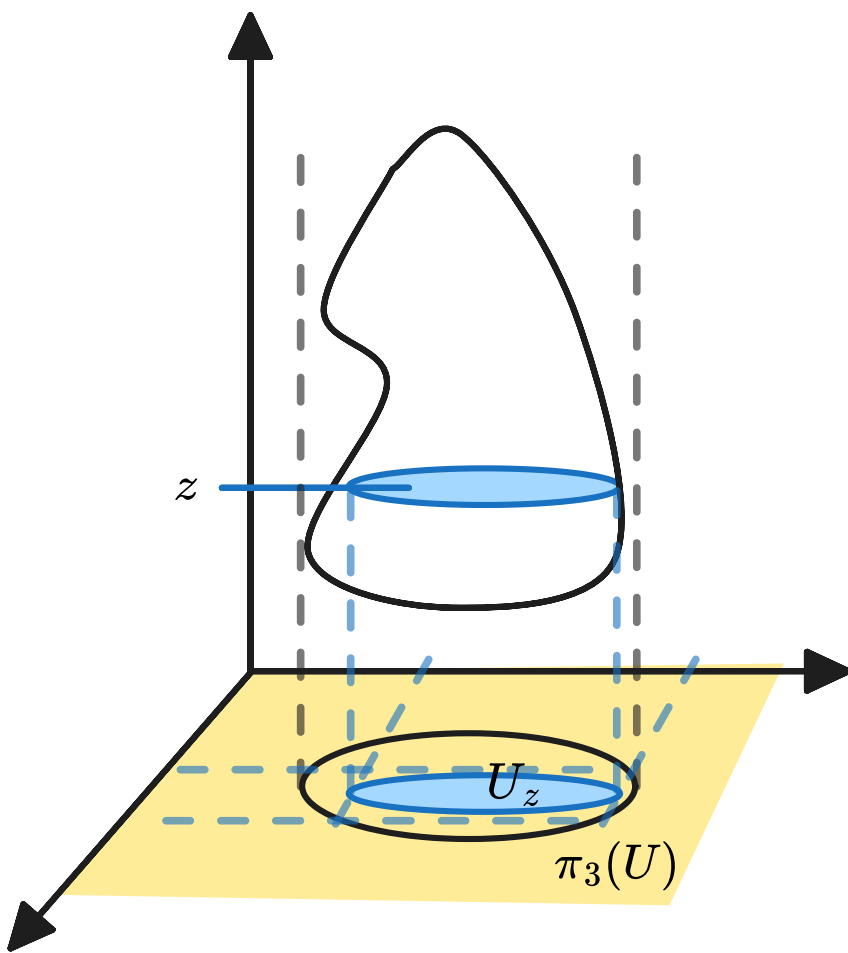
- Consider

$$[0, \epsilon] \times \left[0, \frac{1}{\epsilon}\right] \times \left[0, \frac{1}{\epsilon}\right]$$

 **(Loomis-Whitney)** If $U \subseteq \mathbb{R}^3$ is open then

$$\mu_{\mathbb{R}^3}(U) \leq \prod_{j \in \{1,2,3\}} \sqrt{\mu_{\mathbb{R}^2}(U)}$$





$$\mu_{\mathbb{R}^3}(U) = \int_{z \in \mathbb{R}} \mu_{\mathbb{R}^2}(U_z)$$

- We have

$$\mu_{\mathbb{R}^2}(U_z) \leq \mu_{\mathbb{R}^3}(\pi_3(U))$$

- Also

$$\mu_{\mathbb{R}^2}(U_z) \leq \mu_{\mathbb{R}}(\pi_1(U_z)) \mu_{\mathbb{R}}(\pi_2(U_z))$$

where

$$\mu(\pi_1(U)) = \int_{z \in \mathbb{R}} \mu(\pi_1(U_z))$$

⊗ **Philosophical question:** If we have two bounds for something, what should we do?

1. Pick one bound
2. Use both $f(x) \leq \boxed{1}^\alpha \boxed{2}^{1-\alpha}$
3. Wait and try to figure out for each x which bound is better for $f(x)$.

Choosing (2) in this case we have

$$\begin{aligned} \mu_{\mathbb{R}^3}(U) &= \int_{z \in \mathbb{R}} \mu_{\mathbb{R}^2}(U_z) \\ &\leq \int_{z \in \mathbb{R}} \mu_{\mathbb{R}^3}(\pi_3(U))^\alpha (\mu_{\mathbb{R}}(\pi_1(U_z)) \mu_{\mathbb{R}}(\pi_2(U_z)))^{1-\alpha} \end{aligned}$$

- How should we choose α ?

CMI March '25 - Mini course on Geometric measure theory

✧ Abstract

There will be a series of seven lectures on Geometric measure theory at CMI from 10 March to 21 March 2025.

The first five lectures will be given by Dr. Aswin Govindan Sheri on 10, 11, 13, 14, and 17 March 2025. These lectures will be in hybrid mode.

The last two lectures will be given by Prof. Malabika Pramanik and will be online only.

Date, Time and Venue:

- March 10, 13, 14, and 17, 2025 at 3:30 PM in Seminar Hall
- March 11, 2025 at 5:00 PM in Seminar Hall

Zoom Meeting details (Dr. Aswin Govindan Sheri)

<https://cmi-ac-in.zoom.us/j/84329164349>

Meeting ID: 843 2916 4349

Passcode: 165489

- March 20 and 21, 2025 at 9:00 PM (online talks by Prof. Malabika Pramanik)

Zoom Meeting details (Prof. Malabika Pramanik)

<https://cmi-ac-in.zoom.us/j/89698106713>

Meeting ID: 896 9810 6713

Passcode: 880632

First Five lectures by Dr Sheri would cover the following topics:

Hausdorff measure and dimension, Examples of various types of constructions, Fourier transform and it's basic properties, Frostman's lemma, energy integrals, it's reformulation using Fourier inversion, Distance set conjecture

The mini-course on Geometric Measure Theory will conclude with two lectures on 20, 21 March 2025 by Malabika Pramanik. These lectures aim to showcase the applications of the foundational concepts introduced by Dr. Sheri, while also providing the audience with a glimpse into active areas of research in the field.

The first lecture will explore problems related to pattern recognition, demonstrating how geometric measure theory offers valuable tools for identifying and analyzing patterns in various mathematical and real-world contexts. The second lecture will focus on variants of distance problems, highlighting their significance and complexity in geometric and analytic settings.

These sessions promise to offer both depth and accessibility, intended for participants with a range of mathematical backgrounds.

Prerequisites: Basic course on measure theory

Lecture series abstract: https://drive.google.com/file/d/1Wf-ZSvyd8ZqIDT_2KTBFWuh-bGSuCF9C/view?usp=sharing

AN INTRODUCTION TO GEOMETRIC MEASURE THEORY

ASWIN GOVINDAN SHERI

1. LECTURE SERIES OUTLINE

The purpose of the lecture series is to introduce the students to some of the basic concepts in geometric measure theory. Towards the end of the series, we will also touch upon the topic of distance sets, an active research area in the field. Topics to be covered are given as follows:

- Hausdorff measure and dimension [2, §4.1 - §4.9]
- constructions of various sets; Cantor sets in $\mathbb{R}$ and $\mathbb{R}^n$, and other self-similar sets [2, §4.10 - §4.15], [3, §8]
- energy integrals and Frostman's lemma [2, §8], [3, §2.5]
- Fourier transform, reformulation of the energy integral [2, §12]
- distance sets and Falconer's distance set conjecture [2, §12], a brief sketch of the partial result for the case $\dim A > \frac{n+1}{2}$, $A \subset \mathbb{R}^n$ [3, Theorem 4.6]

Reference textbooks: *Geometry of Sets and Measures in Euclidean Spaces* [2], *Fourier analysis and Hausdorff dimension* [3], *The geometry of fractal sets* [1]

REFERENCES

- [1] K. J. Falconer. *The geometry of fractal sets*, volume 85 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1986.
- [2] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.
- [3] P. Mattila. *Fourier analysis and Hausdorff dimension*, volume 150 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2015.

Email address: agsheri@iiserbpr.ac.in

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH
BERHAMPUR



#people Dr. Aswin Govindan Sheri



I am a post doctoral fellow at the department of Mathematics, [IISER Berhampur](#). Previously, I taught at [National Institute of Technology, Calicut](#) as an ad-hoc faculty. I graduated with a Ph.D. in mathematics from the [University of Edinburgh](#) under [Dr. Jonathan Hickman](#).

- <https://sites.google.com/view/agovindansheri/>
- MS thesis: [Takeya Sets in Harmonic Analysis](#)
- PhD: [On certain geometric maximal functions in Harmonic analysis](#)

- [Lecture 1 by Dr Aswin Govindan Sheri](#)
- [Lecture 2 by Dr Aswin Govindan Sheri](#)

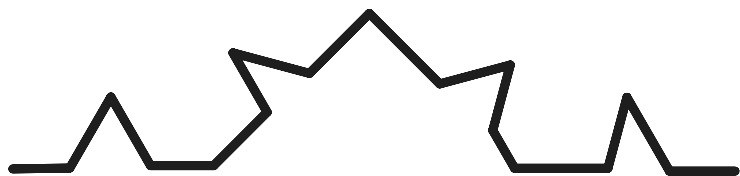
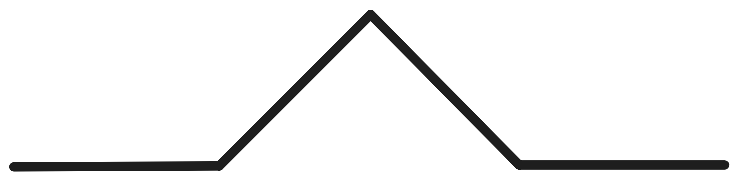
- [Lecture 3 by Dr Aswin Govindan Sheri](#)
- [Lecture 4 by Dr Aswin Govindan Sheri](#)

Middle third Cantor set

$$E := \bigcap_{k \geq 1} E_k$$

- self-similar
- fine structure, details at every scale
- defined recursively
- "complicated" local geometry
- $\mu(E) = 0$ "thin set"

Von Koch curve



$$E_k$$

us a union of 4^k line segments of length

$$\frac{1}{3^k}$$

$$E := \lim_{k \rightarrow \infty} E_k$$

- self-simmilar
- recursive definition

Hausdorff measure through Carathéodory construction

 **Definition.** generalized Hausdorff measure

Let $\mathcal{F}$ be a family of subsets of $\mathbb{R}^d$ along with

$$\mathcal{C} : \mathcal{F} \rightarrow [0, \infty]$$

such that

- for each $\delta > 0$ there exists $E_i \in \mathcal{F}, i \geq 1$ such that

$$d(E_i) \leq \delta, \mathbb{R}^d = \bigcup_{i \geq 1} E_i$$

- for each $\delta > 0$ there exists $E \in \mathcal{F}$ such that

$$d(E) \leq \delta, \mathcal{C}(E) \leq \delta$$

where $d(E)$ is diameter of E . Then for $A \subset \mathbb{R}^d, \delta > 0$ we define

$$\psi_\delta^{\mathcal{C}}(A) := \inf \left\{ \sum_{i \geq 1} \mathcal{C}(E_i)^d \middle| A \subset \bigcup_{i \geq 1} E_i, E_i \in \mathcal{F}, d(E_i) < \delta \right\}$$

The **generalized Hausdorff measure** based on $\mathcal{C}$ is

$$\psi^{\mathcal{C}}(A) := \sup_{\delta > 0} \psi_\delta^{\mathcal{C}}(A)$$

Proposition:

- ψ_δ is monotonic, countably sub-additive but not additive

$$\bullet \hspace{10em} U := B(0, \delta)$$

then

$$\psi_\delta(U) = \psi_\delta(\bar{U}) = \psi_\delta(\partial U) = \delta^s$$

- $\psi_\delta(\emptyset) = 0$

 The measure ψ is a Borel measure and if $\mathcal{F}$ consists of Borel sets then H is Borel regular.

Use

 **(Caratheodory criterion)** If μ is an outer measure on $\mathbb{R}^d$ then μ is Borel measurable iff

$$\mu(A \sqcup B) = \mu(A) + \mu(B)$$

when $A, B \subseteq \mathbb{R}^d$ and $d(A, B) > 0$.

- *(verifying ceriterion)*

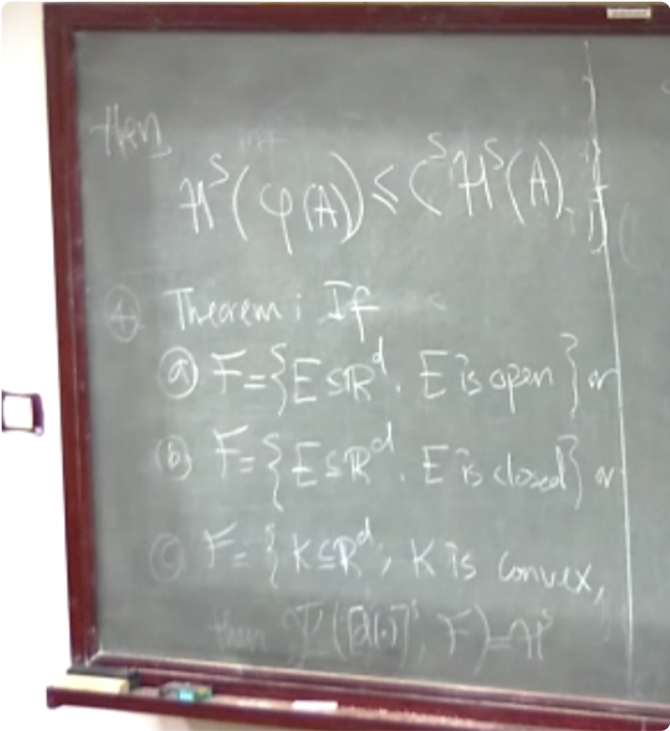
examples

$\mathcal{F}$	$\mathcal{C}$	ψ
$A \in 2^{\mathbb{R}^d}$	$d(A)^s$ $s \in [0, \infty]$	s -Hausdorff measure H^s
	1	H^0 is counting measure

$\mathcal{F}$	$\mathcal{C}$	ψ
$\left\{ R = \prod_i [a_i, b_i] \mid a_i < b_i \right\}$ in $\mathbb{R}^d$	$\text{vol}(R)$	Lebesgue measure $\mu_{\mathbb{R}^d}$ and we can see $\mu_{\mathbb{R}^d} = cH^d$ where $c = \mu_{\mathbb{R}^d}(B_1(0))$
all balls in $\mathbb{R}^d$	$d(E)^t$ for $t \in [0, \infty)$	S^t where it is clear that $H^t(A) \leq S^t(A)$


Exercise

$$S^t(A) \leq 2^t H^t(A)$$



Heat equation

📌 Definition. Heat equation on a Riemannian manifold

A function

$$u : I \times M \rightarrow \mathbb{R}$$

solves the **heat equation** if

$$\frac{\partial u}{\partial t} = \Delta_g u$$

on a Riemannian manifold (M, g)

- $t \geq 0$ usually, so $I = [0, \infty)$
- Model for *diffusion* processes.
- **Scaling**
 - If u solves

📌 Definition. Heat equation on a Riemannian manifold

A function

$$u : I \times M \rightarrow \mathbb{R}$$

solves the **heat equation** if

$$\frac{\partial u}{\partial t} = \Delta_g u$$

on a Riemannian manifold (M, g)

on $\mathbb{R}^n$ then

$$(x, t) \mapsto \beta u(\alpha x, \alpha^2 t)$$

also solves it.

- Finite speed of propagation
 - Fundamental solution on $\mathbb{R}^n$

$$\frac{1}{\sqrt{(4\pi t)^n}} e^{-|x|^2/4t}$$

[1]

📌 Definition. An **ancient solution** to a *parabolic equation* is a solution defined for all $t \in (-\infty, T]$ where $T \in \mathbb{R} \cup \infty$.

📖 Keywords

"Blowup analysis of non-linear parabolic gives ancient solutions", "singularities"

1. Heat equation
2. semi-linear heat equation

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \quad p > 1$$

3. Curve shortening flow & mean curvature flow
4. 2 dimensional Ricci flow (much simpler than 3 dimensional counterpart)

examples of ancient solutions

Example

$$u(x, t) = \phi(x)$$

where ϕ is harmonic which means

$$\Delta u = 0$$

Example

$$u(x, t) = e^{x_1 + t}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n, t \in \mathbb{R}$

Then

$$u_t = e^{x_1 + t} = u_{xx} = \Delta u$$

Example

$$u(t, x) = \frac{1}{\sqrt{(4\pi(T-t))^n}} e^{-|x|^2/4(T-t)}$$

for $t < T$

uniqueness theorems for ancient solutions for Heat equation

≡ **(Yau, 1975)** Let M be a **complete non-compact** Riemannian manifold of non-negative Ricci curvature (Ricci(M) ≥ 0). If $u \geq 0$ is a harmonic function on M , that is $\Delta_g u = 0$ on M , then u is **constant**.

whose proof is based on the *estimate*

≡ **(Cheng-Yau, 1975)** Let M be a **complete non-compact** Riemannian manifold of non-negative Ricci curvature (Ricci(M) ≥ 0). If $u \geq 0$ is a harmonic function on M , that is $\Delta_g u = 0$ on M , then

$$\frac{\|\text{grad}(u)\|(x)}{u(x)} \leq \frac{C_n}{R}$$

for all $x \in B_R(x_0)$ and $x_0 \in M$ where $C_n \in \mathbb{R}$ only depends on $\dim M$.

- This estimate reminds us of the *gradient estimate* for harmonic functions

$$\sup_{B_R(x_0)} |\mathfrak{D}u| \leq \frac{C_n}{R} \sup_{B_R(x_0)} |u|$$

? Question

Does the analogue of Yau's result hold for ancient solution of the heat equation: a $u \geq 0$ such that

$$u_t = \Delta_g u$$

on complete non-compact Riemannian manifold $\times (-\infty, T)$?

The answer is **nooooo!** We have seen the counterexample precisely:

Example

$$u(x, t) = e^{x_1 + t}$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n, t \in \mathbb{R}$

Then

$$u_t = e^{x_1 + t} = u_{xx} = \Delta u$$

But this is *not* really an analogue because t has no meaning in the parabolic(?) setting

(Souplet, Zang, 2007?) Let M be a **complete non-compact** Riemannian manifold of non-negative Ricci curvature ($\text{Ricci}(M) \geq 0$) and let

$$u : M \times (-\infty, T) \rightarrow \mathbb{R}$$

be a solution of heat equation

$$u_t = \Delta_g u$$

then

1. If $u \geq 0$ and

$$u(x, t) = e^{o(d(x, x_0) + \sqrt{|t|})} \text{ as } d(x, x_0) \rightarrow \infty$$

then u is a constant on M .

2. If

$$u(x, t) = o(d(x, x_0) + \sqrt{|t|}) \text{ as } |x| \rightarrow \infty$$

then $u \equiv 0$.

Let M be a complete non-compact Riemannian manifold of non-negative Ricci curvature ($\text{Ricci}(M) \geq 0$) and let

$$u : M \times (-\infty, T) \rightarrow \mathbb{R}$$

be a solution of heat equation

$$u_t = \Delta_g u$$

then

$$\frac{\|\text{grad}(u)\|}{u} \leq c_n \left(\frac{1}{R} + \frac{1}{\sqrt{T}} \right) \left(1 + \ln \frac{L}{u} \right)$$

which holds on

$$Q_{R,T}(x_0) := B_R(x_0) \times [t_0 - T, t_0]$$

and

$$u \leq L \text{ on } Q_{2R,2T}(x_0)$$

(For $\mathbb{R}^n$, we use the *maximal principle*!)

proof of Souplet-Zang from the heat estimate on $\mathbb{R}^n$

- Assume $L = 1$ that is $u \leq 1$ on $Q_{2R,2T}$ where $T = 4R^2$

- $$\ln(u + 1) \leq o(|x| + \sqrt{t})$$

references

- In [UIMP 2015 Summer Course](#) "*Liouville theorems for parabolic equations with emphasis on geometric flows*"
Slides [1](#) [2](#) [3](#)
- [0.pdf](#)
- [rnoti-p467.pdf](#)

Masoud Khalkhali - Spectral geometry

[Spectral Geometry](#)

[Spectral Geometry -Masoud Khalkhali](#)

Weyl's asymptotic law

Let $\Omega \subset \mathbb{R}^n$ is open with piecewise smooth boundary and with **Laplacian** $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ let

$$\begin{aligned}\Delta\varphi &= \lambda\varphi \\ \varphi\Big|_{\partial\Omega} &= 0\end{aligned}$$

be the Helmholtz equation with Dirichlet boundary condition.

- the **spectrum**

$$\begin{aligned}\text{spec}(\Omega) &:= \sigma_p(\Delta : L^2(\Omega) \rightarrow L^2(\Omega)) \\ &= \{\lambda \mid \Delta\varphi = \lambda\varphi\}\end{aligned}$$

is known to be **discrete** with

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \rightarrow \infty$$

- the **eigenfunctions** of Δ form an **orthonormal basis** of $L^2(\Omega)$ (when normalized)

 **(Weyl's asymptotic law, 1911)** Let $\Omega \subset \mathbb{R}^n$ is open with piecewise smooth boundary. Then element λ_k of spectrum of Ω grow like

$$\lambda_k \sim \frac{(2\pi)^2}{(\omega_n \text{vol}(\Omega))^{2/n}} k^{2/n}$$

as $k \rightarrow \infty$ where $\omega_n = \text{vol}(D^n)$ is volume of the unit ball $D^n \subset \mathbb{R}^n$.

So the spectrum determines the volume!

Definition. Isospectral open sets of $\mathbb{R}^n$

Two open sets $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ are called **isospectral** if

$$\text{spec}(\Omega_1) = \text{spec}(\Omega_2)$$

and the multiplicity of each eigenvalue are also equal, that is for each $\lambda \in \text{spec}(\Omega_1) = \text{spec}(\Omega_2)$ we have

$$\dim \ker(\Delta\Big|_{L^2(\Omega_1)} - \lambda \text{Id}) = \dim \ker(\Delta\Big|_{L^2(\Omega_2)} - \lambda \text{Id})$$

Isometric open sets are isospectral!

questions on spacetimes

Birkhoff's theorem: any non-flat vacuum spacetime which is "spherically symmetric" is locally isometric to Schwarzschild spacetime

https://annegretburtscher.wordpress.com/wp-content/uploads/2019/11/willemvanoosterhout_bscthesis_2019.pdf

8.4.6 SPHERICAL SYMMETRY

Define (M, g) as *spherically symmetric* iff $\mathcal{G}M$ contains a subgroup $\mathcal{H}$ with the following two properties: $\mathcal{H}$ is isomorphic to the group $\mathcal{SO}^3$ of proper rotations of Euclidean 3-space, and each orbit of $\mathcal{H}$ is spacelike and at most 2-dimensional.

- (a) Show that then an orbit diffeomorphic to the 2-sphere has the intrinsic geometry of a 2-sphere in Euclidean space.
- (b) Show that then no orbit of $\mathcal{H}$ is one-dimensional.
- (c) Show that each spacetime defined in Section 1.4 is spherically symmetric.
- (d) Suppose (M, g) is Einstein–de Sitter spacetime and $\mathcal{H}$ is one of the subgroups with the above properties. Show there are 0-dimensional orbits, and that all these orbits lie on one integral curve of ∂_4 . (Roughly: each such orbit is a “center of rotation.”)
- (e) Suppose (M, g) is a normal Schwarzschild spacetime or a Schwarzschild black hole, and let $\mathcal{H}$ be as above. Show there is no 0-dimensional orbit (no “center”; in spherically symmetric models of stable stars one replaces part of a normal Schwarzschild spacetime with a different spacetime, using some appropriate matter model; then there is a timelike geodesic—the history of the center of the star—each point on which is a zero-dimensional orbit).
- (f) Show by an example that there exists a spherically symmetric spacetime such that each orbit of $\mathcal{H}$ is diffeomorphic to a projective plane. (This case is sometimes excluded by definition.)
- (g) Show by an example there is a spacetime such that $\mathcal{G}M$ is isomorphic to $\mathcal{SO}^3$ but each orbit is 3-dimensional. (Hint: consider $\mathcal{SO}^3 \times \mathbb{R} = M$.)

8.4.7 SCHWARZSCHILD AND KRUSKAL SPACETIMES

Let (M, g) be a Ricci-flat, spherically symmetric spacetime that is not flat.

- (a) Show (M, g) is locally isometric to a Kruskal spacetime.
- (b) Give a global version of (a).

Title: **When does a vector field kill area?**

Date: 22-Feb-25

Abstract: A Hamiltonian vector field of a given smooth function H on the plane $\mathbb{R}^2$ is the 90-degree rotation of its gradient vector field $\text{grad}(H)$. With this simple definition, we can picture the flow of such a vector field: it must be along the "curves" where H is constant. Thus, we say the Hamiltonian vector field of H "preserves H ". We also observe such flows preserve area on the plane, which begs the question: Are all area-preserving flows Hamiltonian? We try to answer these questions on the plane and its naive generalizations to the 2-sphere and complex vector spaces. And then, we try to generalize Hamiltonian vector fields to smooth manifolds properly: we understand that the minimal structure needed to define a Hamiltonian vector field is precisely the choice of a symplectic form. In this general setting, we shall discuss many introductory topics (Darboux theorem, Moser's theorem, Noether's theorem) and advanced questions/open problems in the field (Arnold conjecture), motivated through examples.

<https://ggl.link/hamiltonian>

Hamiltonian vector fields in different spaces

$\mathbb{C} = \mathbb{R}^2$	S^2	$\mathbb{C}^n = \mathbb{R}^{2n}$	(M, ω) <i>symplectic manifolds</i>
$J = [i]$	J is the complex structure on the Riemann sphere	$J = \begin{bmatrix} i & & \\ & i & \\ & & \ddots \end{bmatrix}$	not available in general!
$X_H := J\text{grad}(H)$	$X_H := J\text{grad}(H)$	$X_H := J\text{grad}(H)$	$X_H := -\omega^{-1}\text{d}H$ <i>notice the "-" sign!</i>
$X_H H = 0$	$X_H H = 0$	$X_H H = 0$	$X_H H = 0$
<i>area preserving</i> $\mathcal{L}_{X_H}\text{d}x \wedge \text{d}y = 0$	<i>area preserving</i> $\mathcal{L}_{X_H}\lambda_{\text{area}} = 0$	volume preserving, but preserves more...	<i>"k-dimensional area preserving"</i> $\mathcal{L}_{X_H}\omega^k = 0$
Converse? True for $\mathbb{R}^2$ but not true for general <i>open subsets!</i>	Converse? true for S^2 !	Converse for volume preservation is false for $n > 2$, even for linear flows!	Converse is false for $\dim M > 2$! However, it is true that vector fields that preserve the <i>symplectic 2-form</i> are Hamiltonian on a simply connected domain.
Linear maps that are <i>generated</i> by Hamiltonian vector fields are just the area preserving ones $Sp_{\mathbb{R}}(2) = SL_{\mathbb{R}}(2)$	What could be "linear" flows? well isometries $SO_{\mathbb{R}}(3)$ preserve area, so we have one nice Lie group action on the 2-sphere.	$SL_{\mathbb{R}}(2n) \supset Sp_{\mathbb{R}}(2n) \supset U(n)$	symplectic G -actions, Hamiltonian G -actions

on the plane

We want a vector field that *preserves* a given smooth

$$H : \mathbb{R}^2 \rightarrow \mathbb{R}$$

Then

$$\text{grad}(H)$$

does the opposite: its flow goes out of $H^{-1}(c)$ *curves*.

So we should work with the 90 degrees rotation of the gradient, so

$$J\text{grad}(H)$$

where

$$J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

is the clockwise 90 degrees rotation matrix.

The flow of this vector field ϕ^t such that

$$\frac{d}{dt}\phi^t(p) = J\text{grad}(H)_{\phi^t(p)}$$

which quickly implies

$$\frac{d}{dt}H(\phi^t(p)) = dH\left(\frac{d}{dt}\phi^t(p)\right) = \langle \text{grad}(H), J\text{grad}(H) \rangle = 0$$

by chain rule!

Hamiltonian vector fields on plane preserve area

$$X_H = \begin{bmatrix} -\frac{\partial H}{\partial y} \\ \frac{\partial H}{\partial x} \end{bmatrix}$$

$$\text{div}(X_H) = -\frac{\partial}{\partial x} \frac{\partial H}{\partial y} + \frac{\partial}{\partial y} \frac{\partial H}{\partial x} = 0$$

And divergence free vector fields preserve area (volume in $\mathbb{R}^n$)

space.R.n.Vec.volume

Volume change by flows on $\mathbb{R}^n$

Consider a **smooth** flow φ^t on some open subset of $\mathbb{R}^n$, this means

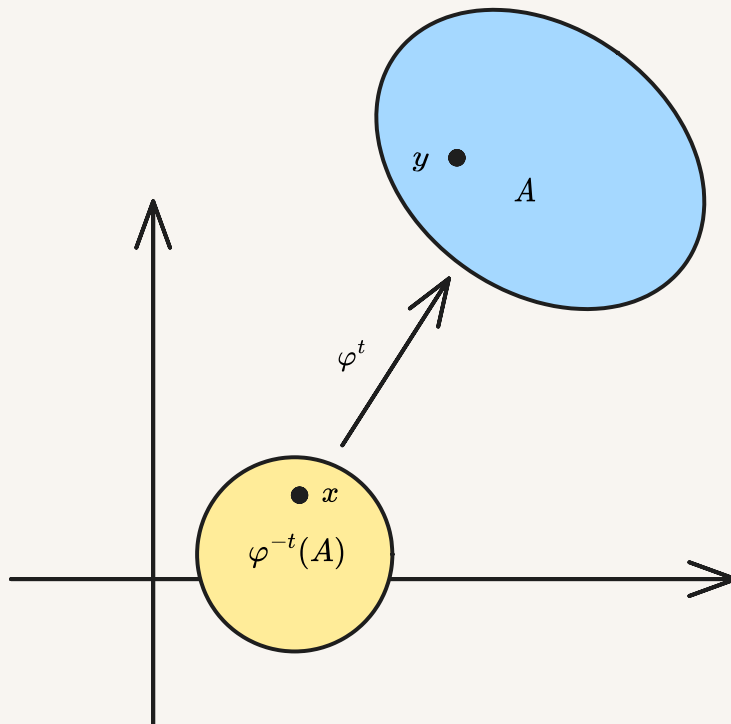
$$\varphi^t : U_t \subseteq U \rightarrow \mathbb{R}^n$$

is an injective local diffeomorphism. The *only* problem is surjectivity!

Then choose $A \subseteq \mathbb{R}^n$ so that it's preimage is not empty, then

$$\varphi^t : \varphi^{-t}(A) \rightarrow A$$

is a diffeomorphism.



We wish to compare volume of A with volume of $\varphi^{-t}(A)$. Thus we calculate volume

- By *change of variables* we get

$$\begin{aligned}\mu(A) &= \int_{y \in A} dy_1 \wedge \dots \wedge dy_n \\ &= \int_{x \in \varphi^{-t}(A)} d\varphi_1^t(x) \wedge \dots \wedge d\varphi_n^t(x) \\ &= \int_{x \in \varphi^{-t}(A)} (\det \mathfrak{D}\varphi^t) dx_1 \wedge \dots \wedge dx_n\end{aligned}$$

- So

$$\mu(\varphi^{-t}(A)) - \mu(A) = \int_{x \in A} (1 - \det \mathfrak{D}\varphi^t) dx_1 \wedge \dots \wedge dx_n$$

- Now

$$\varphi^0 = \text{Id} \implies \det \mathfrak{D}\varphi^0 = 1$$

which means

$$\lim_{t \rightarrow 0} \frac{1}{t} (1 - \det \mathfrak{D}\varphi^t) = \left. \frac{d}{dt} \right|_{t=0} \det \mathfrak{D}\varphi^t$$

- Thus

$$\left. \frac{d}{dt} \right|_{t=0} \mu(\varphi^{-t}(A)) = \int_{x \in A} \left(\left. \frac{d}{dt} \right|_{t=0} \det \mathfrak{D}\varphi^t \right) dx_1 \wedge \dots \wedge dx_n$$

So rate of change of volume boils down to $\left. \frac{d}{dt} \right|_{t=0} \det \mathfrak{D}\varphi^t$.

computing derivative of $\det \mathfrak{D}\varphi^t$ using wedge products

$$\begin{aligned}
\frac{d}{dt} \det(\mathfrak{D}\varphi^t) &= \frac{d}{dt} \left(\frac{\partial \varphi^t}{\partial x_1} \wedge \cdots \wedge \frac{\partial \varphi^t}{\partial x_n} \right) \\
&= \frac{\partial}{\partial x_1} \underbrace{\frac{d}{dt} \varphi^t}_{X(\varphi^t(x))} \wedge \frac{\partial \varphi^t}{\partial x_2} \wedge \cdots \wedge \frac{\partial \varphi^t}{\partial x_n} + \cdots \\
&= \left(\mathfrak{D}X \left(\frac{\partial}{\partial x_1} \varphi^t \right) \right) \wedge \cdots \\
&= \underbrace{\frac{\partial X_1}{\partial x_1} \det \mathfrak{D}\varphi^t + \cdots}_{\det(\mathfrak{D}\varphi^t) \operatorname{div} X}
\end{aligned}$$

computing derivative of $\det \mathfrak{D}\varphi^t$ using Jacobi's formula

From

 (Jacobi's formula)

$$\frac{d}{dt} \det A(t) = (\det A(t)) \operatorname{tr} \left(A(t)^{-1} \frac{d}{dt} A(t) \right)$$

and

$$\begin{aligned}
A'(t) &= B(t)A(t) \\
\implies (\det A)'(t) &= \det(A(t)) \operatorname{tr}(A(t)^{-1} B(t) A(t)) \\
&= \det(A(t)) \operatorname{tr}(B(t))
\end{aligned}$$

along with

$$\begin{aligned}
\frac{\partial}{\partial t} \varphi(t, x) &= X_{\varphi(t, x)} \\
\implies \frac{\partial}{\partial t} \mathfrak{D}\varphi(t, x) &= \mathfrak{D}X_{\varphi(t, x)} \\
&= \mathfrak{D}_{\varphi(t, x)} X \circ \mathfrak{D}\varphi(t, x)
\end{aligned}$$

we get

$$\begin{aligned}
\frac{d}{dt} \mathfrak{D}\varphi^t &= \mathfrak{D} \frac{d}{dt} \varphi^t = \mathfrak{D}X_{\varphi^t} = \mathfrak{D}X \circ \mathfrak{D}\varphi^t \\
\implies \operatorname{tr}(\mathfrak{D}\varphi^{-t} \circ \mathfrak{D}X \circ \mathfrak{D}\varphi^t) &= \operatorname{tr}(\mathfrak{D}X)
\end{aligned}$$

So we have

$$\begin{aligned}
\frac{d}{dt} \det \mathfrak{D}\varphi_{(t, x)}^t &= \operatorname{tr}(\mathfrak{D}X)_{(\varphi^t(x))} \det \mathfrak{D}\varphi_{(t, x)}^t \\
&= \operatorname{div}(X)_{(\varphi^t(x))} \det \mathfrak{D}\varphi_{(t, x)}^t
\end{aligned}$$

- At $t = 0$, $\varphi^0 = \operatorname{Id}$ thus we have

$$\left(\frac{d}{dt} \Big|_{t=0} \det \mathfrak{D}\varphi^t \right)(x) = \operatorname{div}(X)_x$$

- In general

$$\det \mathfrak{D}\varphi_{(t, x)}^t = \exp \left(\int_{[0, t]} \operatorname{div}(X)_{\varphi^t(x)} dt \right)$$

[1]

[2]

computing derivative of $\det \mathcal{D}\varphi^t$ using Liouville's formula

[3]

rate of change of volume is determined by integral of divergence

Hence, we have

 For flow

$$\exp(tX) =: \varphi^t$$

of a smooth vector field X on a open subset of $\mathbb{R}^n$, the rate of change of volume of $\phi^{-t}(A)$, $A \subseteq \mathbb{R}^n$ at $t = 0$ is the volume integral of divergence of X , on A

$$\left. \frac{d}{dt} \right|_{t=0} \mu(\varphi^{-t}(A)) = \int_{x \in A} \operatorname{div}(X)_x dx_1 \wedge \cdots \wedge dx_n$$

Thus

- $\operatorname{div}(X) \equiv 0 \iff$ the flow of X is *volume preserving*

using [Reynolds transport theorem - Wikipedia](#)

1. <https://math.stackexchange.com/a/3456775/1290493> ↩

2. <https://mathoverflow.net/a/327773> ↩

3. <https://math.stackexchange.com/a/312623/1290493> ↩

are all area preserving flows Hamiltonian vector fields?

Well,

- $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is *Hamiltonian* for some smooth H
- $\iff$

$$\exists H \in \mathcal{C}^\infty(\mathbb{R}^2) : \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -\frac{\partial H}{\partial y} \\ \frac{\partial H}{\partial x} \end{bmatrix}$$

- $\iff$

$$\exists H \in \mathcal{C}^\infty(\mathbb{R}^2) : dH = X_2 dx - X_1 dy$$

- So we may use facts about 1-forms

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \leftrightarrow \alpha = X_2 dx - X_1 dy$$

- We know **on a simply connected domain** α is closed ($d\alpha = 0$) $\iff \alpha$ is exact $\exists H \in \mathcal{C}^\infty(\mathbb{R}^2) : \alpha = dH$.
- Now

$$d\alpha = \underbrace{\left(-\frac{\partial X_1}{\partial x} - \frac{\partial X_2}{\partial y} \right)}_{-\text{div}(X)} dx \wedge dy$$

- Hence, on a **simply connected domain** flow of X is area preserving $\iff X$ is divergence free $\iff \exists H \in \mathcal{C}^\infty(\mathbb{R}^2)$ such that

$$X = J \text{grad}(H)$$

-  Notice a minus sign pops out in $d\alpha$ above.

a area preserving flow that is *not* globally Hamiltonian on the punctured plane

The 1-form

Definition. Angle form on $\mathbb{R}^2 \setminus \{0\}$

The angle form on $\mathbb{R}^2 \setminus \{0\}$ is defined by

$$\frac{-y dx + x dy}{x^2 + y^2}$$

using the coordinates (x, y) .

is closed but not exact.

So equating this form with $-\alpha$ above we get the vector field

$$X_\theta := \frac{1}{x^2 + y^2} \begin{bmatrix} x \\ y \end{bmatrix}$$

which has zero divergence but does not have a globally defined Hamiltonian.

It is however locally Hamiltonian for the angle function

$$X_\theta = J\text{grad}(\theta)$$

defined on any simply connected subset of $\mathbb{R}^2 \setminus \{0\}$.

Hamiltonian but in different coordinates

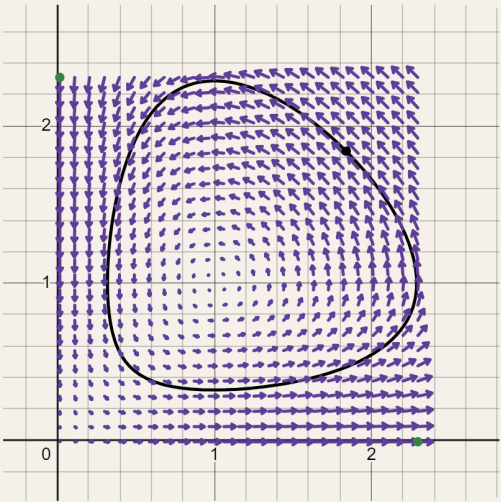
space.f.field.usual Lotka-Volterra

Lotka-Volterra vector field on $\mathbb{R}^2$

 **Definition.** Lotka-Volterra vector field on $\mathbb{R}^2$

$$\begin{bmatrix} \lambda_1 x - \eta_1 xy \\ -\lambda_2 y + \eta_2 xy \end{bmatrix}$$

which looks like



on $\mathbb{R}^2_{>0}$
[1]

for $\lambda_i, \eta_i = 1$

•

$$\text{div} \begin{bmatrix} x - xy \\ -y + xy \end{bmatrix} = 1 - y - 1 + x = x - y$$

Intuition

The flow is not area preserving of course, but it has a "*hidden area preserving*" because the flow is periodic on $\mathbb{R}^2_{>0}$.

• However, it preserves a *different* area form on $\mathbb{R}^2_{>0}$

$$\mathcal{L}_X \left(\frac{dx \wedge dy}{xy} \right) = \text{div} \left(\frac{1}{xy} X \right) = 0$$

• Consider the smooth function for $(x, y) \in \mathbb{R}^2_{>0}$

$$x - \log x + y - \log y$$

- The vector field preserves this function

$$\begin{aligned}\partial_{\begin{bmatrix} x-xy \\ -y+xy \end{bmatrix}}(x - \log x + y - \log y) &= (x - xy) \left(1 - \frac{1}{x}\right) + (-y + xy) \left(1 - \frac{1}{y}\right) \\ &= (x - xy - y + xy) - 1 + y + 1 - x \\ &= 0\end{aligned}$$

- However,

$$\begin{aligned}\text{grad}(x - \log x + y - \log y) &= \begin{bmatrix} 1 - \frac{1}{x} \\ 1 - \frac{1}{y} \end{bmatrix} \\ \implies J\text{grad}(x - \log x + y - \log y) &= \begin{bmatrix} -1 + \frac{1}{y} \\ 1 - \frac{1}{x} \end{bmatrix} = \frac{1}{xy} \begin{bmatrix} x - xy \\ -y + xy \end{bmatrix}\end{aligned}$$

- This shows the vector field is Hamiltonian vector field with the symplectic form $\frac{1}{xy} dx \wedge dy$ on $\mathbb{R}_{>0}^2$.

✓ <https://www.desmos.com/calculator/l0gvaeds5s>

✓ [Lotka-Volter flow](#)

in logarithmic coordinates

- In coordinates

$$\begin{aligned}\mathbb{R}_{>0}^2 &\rightarrow \mathbb{R}_{>0}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} &\mapsto \begin{pmatrix} \log x \\ \log y \end{pmatrix}\end{aligned}$$

we have

Now it is a Hamiltonian system.

1. <https://www.desmos.com/calculator/l0gvaeds5s> ↩

- Lotka-Volterra's vector field
- diagonal action on $\mathbb{C}^n$

sidenote: vector fields that are (locally) Hamiltonian as well as (locally) gradient

<i>object</i>	equivalent to Cauchy-Riemann operator	Cauchy-Riemann operator/equivalent gives 0 which makes the object a	if Cauchy-Riemann operator/equivalent gives 0 then a primitive exists <i>locally</i>
a complex function $f = u + iv$	the Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}} f = \frac{1}{2} \left((u_x - v_y) + i(u_y + v_x) \right)$	Holomorphic function	$\exists g$ $\frac{\partial g}{\partial \bar{z}} = f$
a (complex) differential (1,0)-form $f(z)dz$	the exterior derivative $d(f(z)dz) = -\frac{\partial f}{\partial \bar{z}} dz \wedge d\bar{z}$	closed differential 1-form/Holomorphic 1-form	$\exists g dz$ $dg = f dz$
(real) differential 1-form $\omega = u dx - v dy$ and its Hodge dual $\star \omega = v dx + u dy$	$d\omega = (u_y + v_x) dx \wedge dy$ $d\star \omega = (v_y - u_x) dx \wedge dy$	closed, coclosed (real) differential 1-form	$\exists \alpha, \beta$ (real) 1-forms $d\alpha = \omega, d\beta = \star \omega$
(real) vector field $\bar{f} = \begin{bmatrix} u \\ -v \end{bmatrix}$	$\text{curl} \bar{f} = -v_x - u_y$ $\text{div} \bar{f} = u_x - v_y$	incompressible (area preserving/symplectic) and a solenoid (curl free) vector field	$\exists H, \phi$ such that $J \text{grad}(H) = \bar{f} = \text{grad}(\phi)$

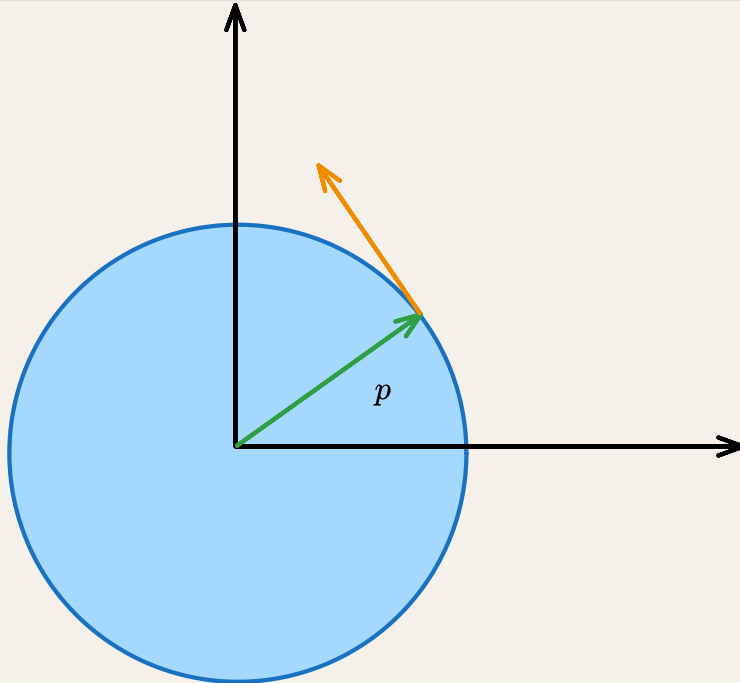
on the 2-sphere

Almost complex structure on S^2 induced from spherical metric and its area form

📌 **Definition.** Almost complex structure on S^2 using cross product on $\mathbb{R}^3$.

The involution

$$J : TS^2 \rightarrow TS^2$$
$$X_p \mapsto p \times X_p$$



where we've identified $T_p S^2 \leq \mathbb{R}^3$.

Because area spanned by the parallelogram is

$$\|X\| \|Y\| \sin(\theta_{X,Y})$$

and *oriented area* is

- supposed to be

$$\begin{aligned} \langle X, Y \rangle &= \|X\| \|Y\| \cos(\theta_{X,Y}) \\ &= \|X\| \|Y\| \sin\left(\theta_{X,Y} + \frac{\pi}{2}\right) \\ &= \langle JX, Y \rangle \end{aligned}$$

- $\langle JX, Y \rangle = \langle p \times X, Y \rangle = \langle p, X \times Y \rangle$

which, again, *should* be the oriented area of $X, Y \in T_p S^2$

- In local coordinates

$$\langle JX, Y \rangle = \left\langle J \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle dx \wedge dy = \left\| \frac{\partial}{\partial x} \times \frac{\partial}{\partial y} \right\| dx \wedge dy$$

so we conclude

$$g_S(JX, Y) = \lambda_S(X, Y)$$

So same as before we define

Definition. Hamiltonian vector field of a smooth function on the 2-sphere $H \in \mathcal{C}^\infty(S^2)$ is the vector field

$$X_H = J \text{grad}(H)$$

- It preserves H because of chain rule, again.
- It preserves area

on complex vector spaces

diagonal action on $\mathbb{C}^2$

space.f.oscillators.harmonic.2

diagonal $T^2 \curvearrowright \mathbb{C}^2$

AKA the dynamics of 2 harmonic oscillators

On the symplectic 4-manifold

$$\left(\mathbb{R}^4, \sum_{i=1}^2 \mathrm{d}p_i \wedge \mathrm{d}q_i\right) = \left(\mathbb{C}^2, \frac{1}{2i} \sum_{i=1}^2 \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i\right)$$

we have the following Hamiltonian vector field and flow

equations	vector field	flow	Hamiltonian
$\begin{aligned} \dot{p}_1 &= -a_1 q_1 \\ \dot{q}_1 &= a_1 p_1 \\ \dot{p}_2 &= -a_2 q_2 \\ \dot{q}_2 &= a_2 p_2 \end{aligned}$ or in more revealing, matrix form $\begin{bmatrix} \dot{p}_1 \\ \dot{q}_1 \\ \dot{p}_2 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} & -a_1 & & \\ a_1 & & & \\ & & -a_2 & \\ & & a_2 & \end{bmatrix} \begin{bmatrix} p_1 \\ q_1 \\ p_2 \\ q_2 \end{bmatrix}$ where the matrix is a block matrix with a scaled $\pi/2$-rotation matrix	$\omega_1 \underbrace{\left(-q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1}\right)}_{X_1} + \omega_2 \underbrace{\left(-q_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial q_2}\right)}_{X_2}$		$\omega_1 \underbrace{\frac{p_1^2 + q_1^2}{2}}_{H_1} + \omega_2 \underbrace{\frac{p_2^2 + q_2^2}{2}}_{H_2}$
$\begin{aligned} \dot{z}_1 &= i\omega_1 z_1 \\ \dot{z}_2 &= i\omega_2 z_2 \end{aligned}$ or $\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} i\omega_1 & \\ & i\omega_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$	$\underbrace{\omega_1 i z_1 \frac{\partial}{\partial z_1}}_{X_1} + \underbrace{\omega_2 i z_2 \frac{\partial}{\partial z_2}}_{X_2} = \omega_1 X_1 + \omega_2 X_2$	$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \xrightarrow{t} \begin{bmatrix} \exp(i\omega_1 t) z_1 \\ \exp(i\omega_2 t) z_2 \end{bmatrix}$	$\omega_1 \underbrace{\frac{z_1 \bar{z}_1}{2}}_{H_1} + \omega_2 \underbrace{\frac{z_2 \bar{z}_2}{2}}_{H_2}$

We may rephrase in terms of the diagonal action on $\mathbb{C}^2$ by $\mathbb{T}^2$

action	generating vector field	moment map
diagonal $\mathbb{T}^2$ -action on $\mathbb{C}^2$ <i>with both frequencies = 1</i> given by $\mathbb{T}^2 \curvearrowright \mathbb{C}^2$ $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \xrightarrow{t_1, t_2} \begin{bmatrix} \exp(it_1)z_1 \\ \exp(it_2)z_2 \end{bmatrix}$	the generators of the $\mathbb{T}^2$ -action are the vector fields X_1, X_2 with $[X_1, X_2] = 0$	the H_1, H_2 above with $\{H_1, H_2\} = 0$ give with moment map $(H_1, H_2) : \mathbb{C}^2 \rightarrow \mathbb{R}^2$ $(z, z_2) \mapsto \frac{1}{2}(z_1\bar{z}_1, z_2\bar{z}_2)$ of the Hamiltonian $\mathbb{T}^2$ -action
from the homomorphism $S^1 \rightarrow \mathbb{T}^2 \curvearrowright \mathbb{C}^2$ $t \mapsto l_1t + l_2t$ for $l_1, l_2 \in \mathbb{Z}$ we get a S^1 -action on $\mathbb{C}^2$ with <i>integer frequencies</i>	the generator is $l_1X_1 + l_2X_2$	
$\mathbb{R} \rightarrow \mathbb{T}^2$	$\omega_1X_1 + \omega_2X_2$	
$\mathbb{T}^2 \rightarrow \mathbb{T}^2$		

restricting to the invariant 3-sphere

We restrict to the 3-sphere

$$S_h^3 = \{(p_1, q_1, p_2, q_2) \in \mathbb{R}^4 | p_1^2 + q_1^2 + p_2^2 + q_2^2 = 2h\} \cong_{\text{Man}} S^3$$

which is invariant under the flow of $\omega_1X_1 + \omega_2X_2$ and has orbits of $H_1 = h$. On this sphere, we have the invariant tori

$$T_{h_1, h_2}^2 := \{(p_1, q_1, p_2, q_2) \in \mathbb{R}^4 | p_1^2 + q_1^2 = 2h_1, p_2^2 + q_2^2 = 2h_2\}$$

with $\omega_1h_1 + \omega_2h_2 = h$.

The projection

$$S^3 \setminus \{(0, 0, 0, 1)\} \rightarrow \mathbb{R}^3$$

$$(x, y, z, w) \mapsto (X, Y, Z) := \frac{1}{1 - w}(x, y, z)$$

maps

- the orbit

$$(p_1, q_1, 0, 0) \mapsto (p_1, q_1, 0)$$

where $p_1^2 + q_1^2 = 2h_1$, the image is thus a circle in the x-y plane in $\mathbb{R}^3$

- the orbit

$$(0, 0, p_2, q_2) \mapsto \left(0, 0, \frac{p_2}{1 - q_2}\right)$$

whose image is the z-axis

- $$T_{h_1,h_2}^2 \mapsto \left\{ (X,Y,Z) \middle| X^2 + Y^2 = \frac{2h_1}{(1-w)^2}, Z^2 = \frac{2h_2 - w^2}{(1-w)^2}, w \neq 1 \right\}$$

which should be a torus in $\mathbb{R}^3$

-

restricting to the invariant tori: a torus translation

The subset

$$\begin{aligned} T_{a,b}^2 &:= \{(p_1,q_1,p_2,q_2) \in \mathbb{R}^4 | p_1^2 + q_1^2 = 2a, p_2^2 + q_2^2 = 2b\} &\subseteq (\mathbb{R}^2 \setminus \{0\})^2 \\ S_h^3 &:= \{(p_1,q_1,p_2,q_2) \in \mathbb{R}^4 | p_1^2 + q_1^2 + p_2^2 + q_2^2 = 2h\} &\subseteq \mathbb{R}^4 \setminus \{0\} \end{aligned}$$

which is preserved by the flow of $\omega_1 X_1 + \omega_2 X_2$.

Thus $(\mathbb{R}^2 \setminus \{0\})^2$ has a torus fibration



coordinates	X_1	$\omega_1 X_1 + \omega_2 X_2$	flow
On $\mathbb{R}^4$ we have (p_1,q_1,p_2,q_2)	$-q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1}$	$\omega_1 \underbrace{\left(-q_1 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial q_1}\right)}_{X_1} + \omega_2 \underbrace{\left(-q_2 \frac{\partial}{\partial p_2} + p_2 \frac{\partial}{\partial q_2}\right)}_{X_2}$	
On $(\mathbb{R}^2 \setminus \{0\})^2$ we have the coordinates $r_1^2 = p_1^2 + q_1^2$ θ_1 $r_2 = p_2^2 + q_2^2$ θ_2	$\frac{\partial}{\partial \theta_1}$	$\omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2}$ which is a torus translation $T_{a,b}^2$	

Flow of a general vector field

$$\omega_1 \frac{\partial}{\partial \theta_1} + \omega_2 \frac{\partial}{\partial \theta_2}$$

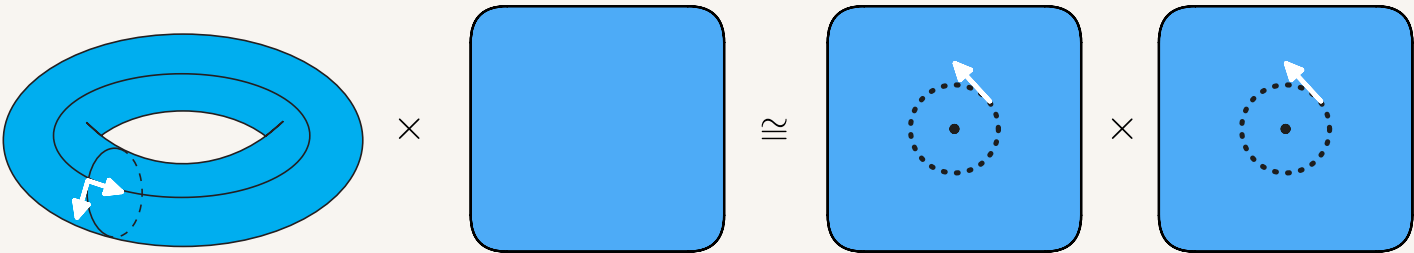
is given by

$$\exp\left(t\left(\omega_1\frac{\partial}{\partial\theta_1}+\omega_2\frac{\partial}{\partial\theta_2}\right)\right)\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}=\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}+t\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}$$

whose flows have properties

if	flow is	which gives a	that is coming from
$\omega_1/\omega_2\in\mathbb{Q}$	periodic	S^1 -action on T^2	composition $S^1\hookrightarrow T^2\curvearrowright T^2$
$\omega_1/\omega_2\in\mathbb{R}\setminus\mathbb{Q}$	not periodic, minimal (orbits are dense)	$\mathbb{R}$ -action on T^2	composition $\mathbb{R}\hookrightarrow T^2\curvearrowright T^2$

decomposition into torus translation $\times$ trivial



$\frac{\partial}{\partial\theta_1}$ $\qquad\qquad\qquad 0\qquad\qquad\qquad \mapsto\qquad\qquad -q_1\frac{\partial}{\partial p_1}+p_1\frac{\partial}{\partial q_1}\qquad\qquad\qquad 0$

$\frac{\partial}{\partial\theta_2}$ $\qquad\qquad\qquad 0\qquad\qquad\qquad \mapsto\qquad\qquad 0\qquad\qquad\qquad -q_2\frac{\partial}{\partial p_2}+p_2\frac{\partial}{\partial q_2}$

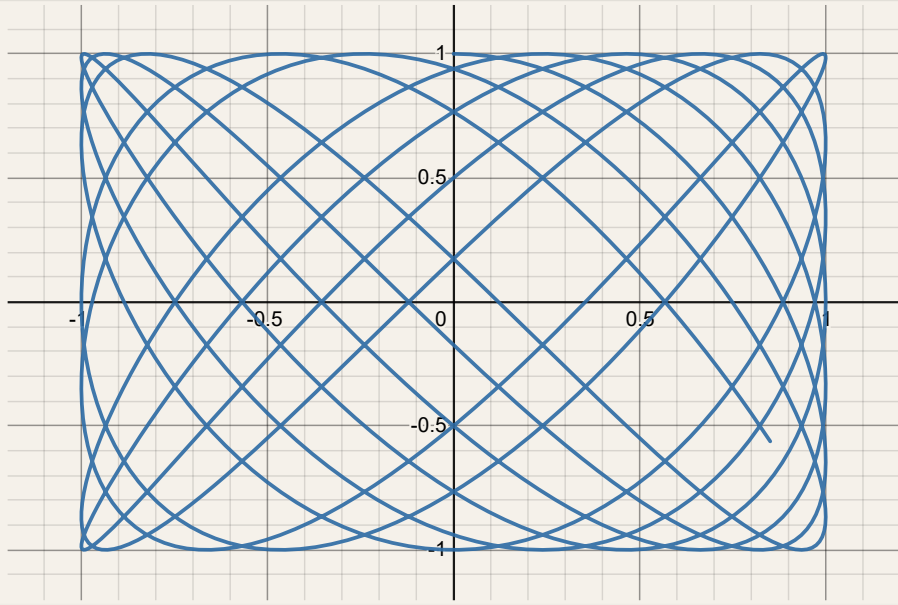
$T^2\curvearrowright T^2\times\mathbb{R}^2\cong_{T^2}(\mathbb{R}^2\setminus\{0\})^2$

$v_1\mapsto\frac{\partial}{\partial\theta_1}\times 0\mapsto X_1$

$v_2\mapsto\frac{\partial}{\partial\theta_2}\times 0\mapsto X_2$

projecting onto (q_1,q_2)

$\{(\sin 45t,\cos 65t)|t\in[0,1]\}$



Projecting $(0, q_1, 0, q_2)$ we have

$$(q_1, q_2) \xrightarrow{t} (\sin(\omega_1 t)q_1, \sin(\omega_2 t)q_2)$$

[1]

rational oscillator when $\omega_1, \omega_2 \in \mathbb{Z}$

A $S^1 \cong U(1)$ -action (or representation) on $\mathbb{R}^4 = \mathbb{C}^2$

irrational oscillator when $\omega_1/\omega_2 \in \mathbb{R} \setminus \mathbb{Q}$

as a Lax pair

as a Lax pair

We define

$$L := \begin{bmatrix} p & q \\ q & -p \end{bmatrix}$$

$$M := \frac{\omega}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

then

$$\begin{aligned} [M, L] &= \frac{\omega}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p & q \\ q & -p \end{bmatrix} - \begin{bmatrix} p & q \\ q & -p \end{bmatrix} \frac{\omega}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \frac{\omega}{2} \begin{bmatrix} -q & p \\ p & q \end{bmatrix} - \frac{\omega}{2} \begin{bmatrix} q & -p \\ -p & -q \end{bmatrix} \\ &= \omega \begin{bmatrix} -q & p \\ p & q \end{bmatrix} \end{aligned}$$

Hence, the oscillator is equivalent to

$$\dot{L} = [M, L]$$

thus giving us conserved quantities

$$\text{tr}(L^2) = \text{tr} \begin{pmatrix} p^2 + q^2 & ? \\ ? & 0 \end{pmatrix} = p^2 + q^2$$

$$L := \begin{bmatrix} p_1 & q_1 \\ q_1 & -p_1 \end{bmatrix} + \begin{bmatrix} p_2 & q_2 \\ q_2 & -p_2 \end{bmatrix} z$$

$$M := \frac{\omega}{2} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$[M, A_1 + A_2 z] = [M, A_1] + [M, A_2] z$$

then

$$\dot{L} = [M, L]$$

should give the equation for isotropic harmonic oscillator.

Now

$$\begin{aligned} yI_2 - A_1 - A_2 z &= \begin{bmatrix} y - p_1 - zp_2 & -q_1 - zq_2 \\ -q_1 - zq_2 & y + p_1 + zp_2 \end{bmatrix} \\ \det(yI_2 - A_1 - A_2 z) &= y^2 - (p_1 + zp_2)^2 - (q_1 + zq_2)^2 \\ &= y^2 - (p_1^2 + q_1^2) - (p_2^2 + q_2^2)z^2 - 2z(p_1p_2 + q_1q_2) \end{aligned}$$

which is a conic section over which

$$(y, z) \mapsto \ker(yI_2 - A(z))$$

defines a line bundle.

[2]

-
1. <https://www.desmos.com/calculator/irnyqunfsm> ↩
 2. <https://www.desmos.com/calculator/2n2puejwvk> ↩

on a symplectic manifold

We want a smooth tensor

$$\omega : T^*M \rightarrow TM$$

that is, smoothly varying (on $p \in M$) linear maps

$$\omega_p : T_p^*M \rightarrow T_p^*M$$

with which we want to define

$$X_H := -\omega^{-1}dH \iff dH(-) = -\omega(X_H, -)$$

as the Hamiltonian vector field of H such that

- **flow of X_H must preserve H**

$$0 = X_H H = dH(X_H) = -\omega(X_H, X_H)$$

$\iff \omega$ is **alternating**, so a 2-form

- and **flow of X_H must preserve ω**

$$0 = \mathcal{L}_{X_H}\omega = \iota_{X_H}d\omega + \underbrace{d\iota_{X_H}\omega}_{-dH} \overset{0}{\rightarrow}$$

which is true when $d\omega = 0$

Thus

📌 **Definition.** A **symplectic manifold** (M, ω) is a **smooth manifold** M with a **closed non-degenerate 2-form** ω .

properties of the symplectic form

Proposition: ω^n defines a volume form

Darboux local coordinates on symplectic manifolds

The Darboux Theorem

Our next theorem is one of the most fundamental results in symplectic geometry. It is a nonlinear analogue of the canonical form for a symplectic tensor given in Proposition 22.7. It illustrates the most dramatic difference between symplectic structures and Riemannian metrics: unlike the Riemannian case, there is no local obstruction to a symplectic structure being locally equivalent to the standard flat model.

Theorem 22.13 (Darboux). *Let (M, ω) be a $2n$ -dimensional symplectic manifold. For any $p \in M$, there are smooth coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ centered at p in which ω has the coordinate representation*

$$\omega = \sum_{i=1}^n dx^i \wedge dy^i. \quad (22.5)$$

We will prove the theorem below. Any coordinates satisfying (22.5) theorem are called **Darboux coordinates**, **symplectic coordinates**, or **canonical coordinates**.

Proof of the Darboux theorem. Let ω_0 denote the given symplectic form on M , and let $p_0 \in M$ be arbitrary. The theorem will be proved if we can find a smooth coordinate chart (U_0, φ) centered at p_0 such that $\varphi^* \omega_1 = \omega_0$, where $\omega_1 = \sum_{i=1}^n dx^i \wedge dy^i$ is the standard symplectic form on $\mathbb{R}^{2n}$. Since this is a local question, by choosing smooth coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$ in a neighborhood of p_0 , we may replace M with an open ball $U \subseteq \mathbb{R}^{2n}$. Proposition 22.7 shows that we can arrange by a linear change of coordinates that $\omega_0|_{p_0} = \omega_1|_{p_0}$.

Let $\eta = \omega_1 - \omega_0$. Because η is closed, the Poincaré lemma (Theorem 17.14) shows that we can find a smooth 1-form α on U such that $d\alpha = -\eta$. By subtracting a constant-coefficient (and thus closed) 1-form from α , we may assume without loss of generality that $\alpha_{p_0} = 0$.

For each $t \in \mathbb{R}$, define a closed 2-form ω_t on U by

$$\omega_t = \omega_0 + t\eta = (1-t)\omega_0 + t\omega_1.$$

Let J be a bounded open interval containing $[0, 1]$. Because $\omega_t|_{p_0} = \omega_0|_{p_0}$ is non-degenerate for all t , a simple compactness argument shows that there is some neighborhood $U_1 \subseteq U$ of p_0 such that ω_t is nondegenerate on U_1 for all $t \in \bar{J}$. Because of

this nondegeneracy, the smooth bundle homomorphism $\hat{\omega}_t: TU_1 \rightarrow T^*U_1$ defined by $\hat{\omega}_t(X) = X \lrcorner \omega_t$ is an isomorphism for each $t \in \bar{J}$.

Define a smooth time-dependent vector field $V: J \times U_1 \rightarrow TU_1$ by $V_t = \hat{\omega}_t^{-1}\alpha$, or equivalently

$$V_t \lrcorner \omega_t = \alpha.$$

Our assumption that $\alpha_{p_0} = 0$ implies that $V_t|_{p_0} = 0$ for each t . If $\theta: \mathcal{E} \rightarrow U_1$ denotes the time-dependent flow of V , it follows that $\theta(t, 0, p_0) = p_0$ for all $t \in J$, so $J \times \{0\} \times \{p_0\} \subseteq \mathcal{E}$. Because $\mathcal{E}$ is open in $J \times J \times M$ and $[0, 1]$ is compact, there is some neighborhood U_0 of p_0 such that $[0, 1] \times \{0\} \times U_0 \subseteq \mathcal{E}$.

For each $t_1 \in [0, 1]$, it follows from Proposition 22.15 that

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=t_1} (\theta_{t,0}^* \omega_t) &= \theta_{t_1,0}^* \left(\mathcal{L}_{V_{t_1}} \omega_{t_1} + \left. \frac{d}{dt} \right|_{t=t_1} \omega_t \right) \\ &= \theta_{t_1,0}^* (V_{t_1} \lrcorner d\omega_{t_1} + d(V_{t_1} \lrcorner \omega_{t_1}) + \eta) \\ &= \theta_{t_1,0}^* (0 + d\alpha + \eta) = 0. \end{aligned}$$

Therefore, $\theta_{t,0}^* \omega_t = \theta_{0,0}^* \omega_0 = \omega_0$ for all t . In particular, $\theta_{1,0}^* \omega_1 = \omega_0$. It follows from Theorem 9.48(c) that $\theta_{1,0}$ is a diffeomorphism onto its image, so it is a coordinate map. Because $\theta_{1,0}(p_0) = p_0 = 0$, these coordinate are centered at p_0 . $\square$

Moser's theorem on volumes

Theorem 2. (Moser) Assume M is a compact, connected and oriented manifold of dimension m without boundary. If α and β are two volume forms such that their total volumes agree, i.e.,

$$(1.40) \quad \int_M \alpha = \int_M \beta,$$

then there is a diffeomorphism u of M satisfying $u^*\beta = \alpha$.

Consequently the total volume is the only invariant of volume-preserving diffeomorphisms.

Proof. We proceed as in the proof of Darboux's theorem and deform the volume form α into β defining

$$\alpha_t = (1-t)\alpha + t\beta, \quad 0 \leq t \leq 1.$$

These forms α_t are volume forms since locally α and β are represented by $\alpha = a(x)dx_1 \wedge \dots \wedge dx_m$ and $\beta = b(x)dx_1 \wedge \dots \wedge dx_m$ with nonvanishing smooth functions a and b , which, by assumption (1.40), must have the same sign. We shall construct a family φ^t of diffeomorphisms satisfying

$$(1.41) \quad (\varphi^t)^* \alpha_t = \alpha, \quad \varphi^0 = id$$

for $0 \leq t \leq 1$, so that the diffeomorphism $u = \varphi^1$ will solve our problem. Since M is compact, connected and oriented we conclude from $\int_M (\beta - \alpha) = 0$ that

$$(1.42) \quad \beta - \alpha = d\gamma$$

for some $(m-1)$ -form γ on M . This is a special case of the de Rham theorem. Since α_t is a volume form we find a unique time-dependent vector field X_t on M solving the linear equation

$$i_{X_t} \alpha_t = -\gamma$$

for $0 \leq t \leq 1$. Denote by φ^t the flow of this vector field X_t satisfying $\varphi^0 = id$. Since M is compact it exists for all t . Since $d\alpha_t = 0$ for volume forms we find, again using Cartan's formula,

$$\begin{aligned} \frac{d}{dt} (\varphi^t)^* \alpha_t &= (\varphi^t)^* \left(L_{X_t} \alpha_t + \frac{d}{dt} \alpha_t \right) \\ &= (\varphi^t)^* \left(d(i_{X_t} \alpha_t) + \beta - \alpha \right), \end{aligned}$$

which vanishes since $d(i_{X_t} \alpha_t) + \beta - \alpha = d(i_{X_t} \alpha_t + \gamma) = 0$ by our choice of the vector field X_t . Therefore (1.41) holds and the proof is finished. ■

references

- Jack Lee's book on smooth manifolds
- <http://staff.ustc.edu.cn/~wangzuoq/Courses/15S-Symp/SympGeom.html>
- Ana Canas da Silver, Lectures on Symplectic Geometry, Lecture Notes in Mathematics 1764, Springer.

Corinna Ulcigrai - Chaotic Properties of Area Preserving Flows

Flows on surfaces are one of the fundamental examples of dynamical systems, studied since **Poincaré**; area preserving flows arise from many physical and mathematical examples, such as the **Novikov model of electrons in a metal, unfolding of billiards in polygons, pseudo-periodic topology**.

In this course we will focus on smooth area-preserving or locally Hamiltonian flows and their ergodic properties. The course will be self-contained, so we will define basic ergodic theory notions as needed and no prior background in the area will be assumed.

The course aim is to explain some of the many developments happened in the last decade. These include the full classification of generic mixing properties (mixing, weak mixing, absence of mixing) motivated by a conjecture by Arnold, up to very recent rigidity and disjointness results, which are based on a breakthrough adaptation of ideas originated from Marina Ratner's work on unipotent flows to the context of flows with singularities.

We will in particular highlight the role played by shearing as a key geometric mechanism which explains many of the chaotic properties in this setup. A key tool is provided by Diophantine conditions, which, in the context of higher genus surfaces, are imposed through a multi-dimensional continued fraction algorithm (Rauzy-Veech induction): we will explain how and why they appear and how they allow to prove quantitative shearing estimates needed to investigate chaotic properties.

- https://www.zora.uzh.ch/id/eprint/194106/1/Ulcigrai2020_Article_SlowChaosInSurfaceFlows.pdf
- <https://homepage.mi-ras.ru/~snovikov/74.pdf>

Lecture 2

 **Definition.** Let $\Sigma \subseteq S$ be an arc transversal to the flow Φ^t on Σ . For $x \in \Sigma$ we let the **minimum return time** $r(x)$ be the minimum $t > 0$ such that $\Phi^t(x) \in \Sigma$, if defined, then

$$\begin{aligned} T : \Sigma &\rightarrow \Sigma \\ x &\mapsto \Phi^{r(x)}(x) \end{aligned}$$

IMPA19 School of Symplectic Topology

[School of Symplectic Topology \(2019\)](#)

Floer homologies

Let

$$(M, \omega)$$

be a symplectic manifold and

$$H : \mathbb{R} \times M \rightarrow \mathbb{R}$$

be smooth.

Then

$$X_H := -\omega^{-1}dH$$

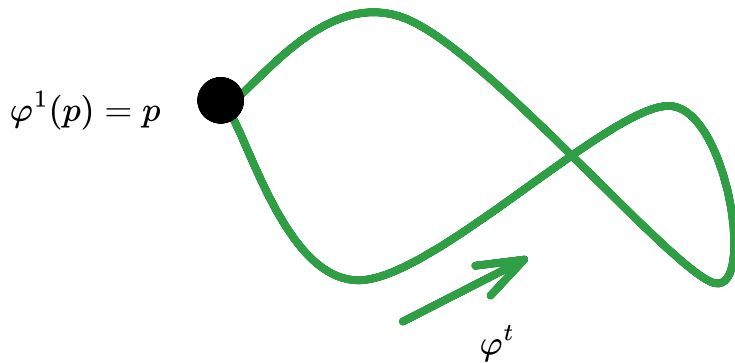
is its Hamiltonian vector field and

$$\frac{d}{dt}\varphi_H^t = X_{\varphi_H^t}$$

is its flow.

Then φ^t is a symplectomorphism

$$(\varphi_H^t)^*\omega = \omega$$



Then 1-periodic orbits are fixed points of φ^1 .

We can reparameterize to get a new

$$H : \frac{\mathbb{R}}{\mathbb{Z}} \times M \rightarrow \mathbb{R}$$

which has same 1-periodic orbits.

⚠ Arnold conjecture

The number of fixed points of the Hamiltonian diffeomorphism is bounded from below

$$|\text{fix}(\varphi_H^1)| \geq \sum_k \dim H_{\text{Mk}}(M, \mathbb{Z}_2)$$

when φ_H^1 is non-degenerate.

Morse homology

📌 Definition. Morse functions

Let M be a compact smooth manifold and

$$f : M \rightarrow \mathbb{R}$$

and $p \in \text{crit}(f)$. Then we have a symmetric bilinear form

$$H_p : T_p M \times T_p M \rightarrow \mathbb{R}$$

The critical point p is **non-degenerate** if H_p is a non-degenerate bilinear form.

The function f is a **Morse function** if all its critical points are non-degenerate.

📖 **(Morse lemma)** For a Morse function $f \in \mathcal{C}^\infty(M)$ and a critical point p , there exists a neighborhood of p and a chart on U on which f looks like

$$f(p) - x_1^2 - x_2^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$$

📌 Definition. Morse index of a critical point

Morse index of a critical point of a Morse function is the i appearing in

📖 **(Morse lemma)** For a Morse function $f \in \mathcal{C}^\infty(M)$ and a critical point p , there exists a neighborhood of p and a chart on U on which f looks like

$$f(p) - x_1^2 - x_2^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_n^2$$

📌 Definition. Smale condition

A pseudo-gradient field satisfies the Smale condition if for all $p, q \in \text{crit}(f)$ the unstable and stable manifold of p, q respectively intersect transversally

$$W^u(p) \pitchfork W^s(q)$$

Their intersection is a manifold in this case.

$$\dim W^u(p) \pitchfork W^s(q) = \text{ind}(p) - \text{ind}(q)$$

📌 Definition. The space of trajectories connecting p to q is

$$\frac{W^u(p) \pitchfork W^s(q)}{x \sim \varphi^s(x)} =: L(p, q)$$

whose dimension is $\text{ind}(p) - \text{ind}(q) - 1$.

📌 Definition. Morse homology

The chain complex with vector spaces

$$C_k := \mathbb{Z}_2 \{p \in \text{crit}(f) \mid \text{ind}(p) = k\}$$

with boundary operator

$$\begin{aligned} \partial_k : \mathcal{C}_k &\rightarrow \mathcal{C}_{k-1} \\ p \mapsto \partial_k(p) &:= \sum_{q \in \text{crit}_{k-1}(f)} (|L(p,q)| \bmod 2) q \end{aligned}$$

where $|L(p,q)| \bmod 2$ counts the number of trajectories mod 2.

$$\sum_{q \in \text{cir}_k} (|L(p,q)| \bmod 2)(|L(q,r)| \bmod 2) = 0$$

Denis Auroux - Fukaya categories and mirror symmetry

【Fukaya范畴与镜像对称】【Columbia】Fukaya categories and mirror symmetry by Denis Auroux_哔哩哔哩_bilibili
http://web.archive.org/web/20191224163140/http://math.columbia.edu/~topology/L1_Luis.pdf

Hofer, Zehnder - Symplectic Invariants and Hamiltonian Dynamics

Helmut Hofer, Eduard Zehnder - Symplectic Invariants and Hamiltonian Dynamics-Birkhäuser Basel (2011).pdf

closed orbits on hypersurfaces

? Question

Does a compact regular H -hypersurface in $(\mathbb{R}^{2n}, \omega_{\text{st}})$ possess a periodic solution of the Hamiltonian vector field X_H ?

Proposition: Let $H, F \in \mathcal{C}^\infty(M)$ and

$$S = H^{-1}(c) = F^{-1}(c')$$

with

$$\mathrm{d}H, \mathrm{d}F \neq 0 \text{ on } S$$

(S is both H, F -regular hypersurface). Then the flows of X_H, X_F are reparameterizations of one another. In particular the the periodic orbits of X_H, X_F are the same.

💡 We have

$$\ker \mathrm{d}F = \ker \mathrm{d}H$$

and

$$\mathrm{d}F = \rho \mathrm{d}H \text{ on } S$$

for $\rho \in \mathcal{C}^\infty(S)$.

- This makes

$$X_F = \rho X_H$$

which implies the two vector fields have same orbits.

🚩 Somehow the hypersurface $S \subset \mathbb{R}^{2n}$ carries a prescribed collection of ω_{st} -orbits!

boundary of compact, strictly convex regions

☰ Assume $S \subset \mathbb{R}^{2n}$ is a smooth boundary of a compact and strictly convex region in $\mathbb{R}^{2n}$. Then S carries a closed ω_{st} -orbit.

Joe Harris - Algebraic Geometry: a first course

Let $k = \bar{k}$ be a field.

twisted cubic

varieties

affine verities

affine n -space	$\mathbf{A}^n(\bar{k}) := \bar{k}^n$
k -points of $\mathbf{A}^n$	$\mathbf{A}^n(k) := \mathbf{A}^n(\bar{k}) \cap k^n = k^n$
	$\text{Gal}(\bar{k}/k) \curvearrowright \mathbf{A}^n(\bar{k})$ $(x_1, x_2, \dots, x_n) \mapsto (x_1^\sigma, x_2^\sigma, \dots, x_n^\sigma)$
	then $\mathbf{A}^n(k)$ is the set fixed pointwise by the Galois group
affine algebraic set	for an ideal $I \subseteq \bar{k}[X_1, \dots, X_n]$ $V_{\bar{k}}(I) := \{p \in \mathbf{A}^n(\bar{k}) \mid f \in I \implies f(p) = 0\}$
k -points of $V_{\bar{k}}(I)$	$V_k(I) := V_{\bar{k}}(I) \cap \mathbf{A}^n(k)$
geometric ideal in k	$I(V_k(I)) := \{f \in k[X_1, \dots, X_n] \mid p \in V_k(I) \implies f(p) = 0\}$ $= I(V_k(I)) \cap k[X_1, \dots, X_n]$
$V_k(I)$ is an affine variety	if $I(V_{\bar{k}}(I))$ is a prime ideal in $\bar{k}[X_1, \dots, X_n]$
affine coordinate ring of an affine variety $V = V_k(I)$	the <i>integral domain</i> $k[V] := \frac{k[X_1, \dots, X_n]}{I(V)}$
function field of an affine variety V	field of fraction of affine coordinate ring $k(V)$
	$\bar{k}[V] := \frac{\bar{k}[X_1, \dots, X_n]}{I(V_{\bar{k}}(I))}$
the evaluation map on affine coordinate ring	$\text{ev}_p : \bar{k}[V] \rightarrow \bar{k}$ which is surjective
dimension of variety	<i>transcendence degree</i> of $\bar{k}(V)$ over $\bar{k}$
maximal ideal of coordinate functions vanishing at a point	$\mathfrak{m}_p(V) := \{f \in \bar{k}[V] \mid f(p) = 0\} = \ker \text{ev}_p$ which is a maximal ideal
cotangent space at $p \in V$	the $\bar{k}$ -vector space $T_p^*V := \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2}$

non-singular or smooth point $p \in V$	$\text{rank} \left[\begin{array}{ccc} \frac{\partial f}{\partial X_1} & \dots & \frac{\partial f}{\partial X_n} \end{array} \right]_p = n - \dim(V)$ $\iff$ $\dim \frac{\mathfrak{m}_p}{\mathfrak{m}_p^2} = \dim V$
local ring of V at p of "germs of functions at p "	$\bar{k}[V]_p := (\bar{k}[V])_{\mathfrak{m}_p}$ whose elements are said to be <i>regular at p</i>

projective varieties

projective n -space	$\bar{k}P^n := P(k^{n+1})$
minimal field for $p \in \bar{k}P^n$	
homogeneous polynomial	
projective algebraic set	$V(I)$
projective variety	if $I(V)$ is a prime ideal in $\bar{k}[X]$
dimension	Choose $\mathbf{A}^n \subset P^n$ such that $V \cap \mathbf{A}^n \neq \emptyset$, then $\dim V := \dim V \cap \mathbf{A}^n$
non-singular/smooth point $p \in V$	if $p \in V \cap \mathbf{A}^n$ is non-singular at p
local ring of V at p	$\bar{k}[V]_p$ is the local ring of $V \cap \mathbf{A}^n$ at p
rational $f : V_1 \rightarrow V_2$ between projective varieties $V_1, V_2 \subset P^n$	$f = (f_0, \dots, f_n) \in \bar{k}(V_1)^n$ such that for every point $p \in V_1$ where all $f_0, \dots, f_1$ are defined $(f_0(p), \dots f_n(p)) \in V_2$
rational is <i>regular</i> at $p \in V_1$	if there is a $g \in \bar{k}(V_1)$ such that gf_i are well defined at p and $(gf_i)(p) \neq 0$ for some p
morphism of projective varieties	a <i>rational</i> map $f : V_1 \rightarrow V_2$ which is <i>regular</i> at every point

curves

 **Definition.** A **projective curve** is a projective variety of dimension one.

Proposition: Let C be a curve and $p \in C$ be a smooth point. Then

$$\bar{k}[C]_p$$

is a DVR.

📌 Definition. Order of functions on curves

Let C be a curve and $p \in C$ be a smooth point. The **normalized valuation** on $\bar{k}[C]_p$ is

$$\begin{aligned} \text{ord}_p : \bar{k}[C]_p &\rightarrow \mathbb{N} \cup \{\infty\} \\ \text{ord}_p(f) &:= \sup\{d \in \mathbb{Z} \mid f \in \mathfrak{m}_p^d\} \end{aligned}$$

and we extend this map to

$$\text{ord}_p : \bar{k}(C) \rightarrow \mathbb{Z} \cup \{\infty\}$$

by $\text{ord}_p(f/g) := \text{ord}_p(f) - \text{ord}_p(g)$. The **order** of $f \in \bar{k}(C)$ at p is $\text{ord}_p(f)$.

A **uniformizer** for C at p is any $t \in \bar{k}(C)$ with $\text{ord}_p(t) = 1$, that is, a generator for ideal $\mathfrak{m}_p$.

Proposition:

- If $\text{ord}_p(f) > 0$ then f has a zero at p .
- If $\text{ord}_p(f) \geq 0$ then f is *regular* (or *defined*) at p , and we may evaluate $f(p)$.
- If $\text{ord}_p(f) < 0$ then f as a pole at p and $f(p) = \infty$.

Proposition: Let C be a smooth curve and $f \in \bar{k}(C)$ with $f \neq 0$. Then

- there are only finitely many points at which f has a pole or zero
- if f has no poles then $f \in \bar{k}$

Proposition: Let C/k be a curve and $t \in k(C)$ be a uniformizer at some non-singular point $p \in C/k$. Then

$$k(C) \geq k(t)$$

is a finite separable extension.

intersection multiplicities

Let $p \in \mathbb{A}^2(k)$ and two curves F, G then

$$\mu_p(F, G) := \dim \frac{k(\mathbb{A}^2)}{\langle F, G \rangle}$$

schemes

commutative ring with 1_R	R
affine scheme	➤ Definition. Spectrum of a commutative ring The set of all prime ideals of a ring is called spectrum of a commutative ring written $\text{Spec}R$ The set of all maximal ideals are written $\text{maxSpec}R$. The subsets of the form $Z(I) := \{\mathfrak{p} \supset I \text{ is a prime ideal of } R\}$ are called Zariski-closed subsets of $\text{Spec}R$ which define a topology on the spectrum of R.
	$\mathcal{O}_{\mathfrak{p}} := \text{field of fractions of } \frac{R}{\mathfrak{p}}$
evaluating $f \in R$ at $\mathfrak{p}$	$f(\mathfrak{p}) \in \mathcal{O}_{\mathfrak{p}}$ that is image under $R \rightarrow \frac{R}{\mathfrak{p}} \rightarrow \mathcal{O}_{\mathfrak{p}}$
zero locus of $S \subset R$	$Z(S) = \{\mathfrak{p} \in \text{Spec}R \mid f \in S \implies f(\mathfrak{p}) = 0\}$

vector bundles on schemes

Let X be a scheme.	
locally free sheaf $\mathcal{F}$ of $\mathcal{O}_X$ -modules of rank r	if there is a open cover $\{U_i\}$ of X such that $\mathcal{F}_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$
vector bundle of rank r on X over a field k	a k -scheme E and a morphism $\pi : E \rightarrow X$...
differentials	$\Omega_{R/S}$

Fulton & Harris - Representation theory

chapter 2

exercise 2.33

exercise 2.37

if $G \rightarrow GL(V)$ is injective then any irrep of G is contained in some tensor power $V^{\otimes n}$

Use character theory.

Consider

$$a_n := \langle \chi_W, \chi_V^n \rangle$$

where both W, V are fixed. Then $f(z) := \sum_{n \geq 1} a_n z^n$ then

$$\begin{aligned} f(z) &= \frac{1}{|G|} \sum_{n \geq 1} \sum_{g \in G} \overline{\chi_W(g)} (\chi_V(g))^n z^n \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \sum_{n \geq 1} (\chi_V(g))^n z^n \\ &= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_W(g)} \left(\frac{\chi_V(g)z}{1 - \chi_V(g)z} \right) \end{aligned}$$

chapter 12

Representations of $\mathfrak{sl}(3, \mathbb{C})$. Part 1.

Quote

As we will see, a number of the basic constructions need to be modified, or at least rethought. There are, however, two pieces of good news that should be borne in mind.

[Fulton Harris Representation theory, page 175](#)

- Let $\mathfrak{h} \leq \mathfrak{sl}(3, \mathbb{C})$ of diagonal matrices. So

$$\mathfrak{h} = \left\{ \begin{bmatrix} a_1 & & \\ & a_2 & \\ & & -a_1 - a_2 \end{bmatrix} : a_1, a_2 \in \mathbb{C} \right\}$$

which is of $\dim = 2$.

-
- Every finite-dim rep of $\mathfrak{sl}(3, \mathbb{C})$ decomposes into

$$\bigoplus_{\alpha} V_{\alpha}$$

where every $H \in \mathfrak{h}$ is (simul) diagonalizable on V_{α} .

-
- E_{ij} has (i, j) -entry = 1 and rest 0. Then $[D, M]$ is a multiple of M for all D if and only if $M = cE_{ij}$.

$$[D, E_{ij}] = (D_i - D_j)E_{ij}$$

- We want

$$\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}_{\alpha}$$

such that

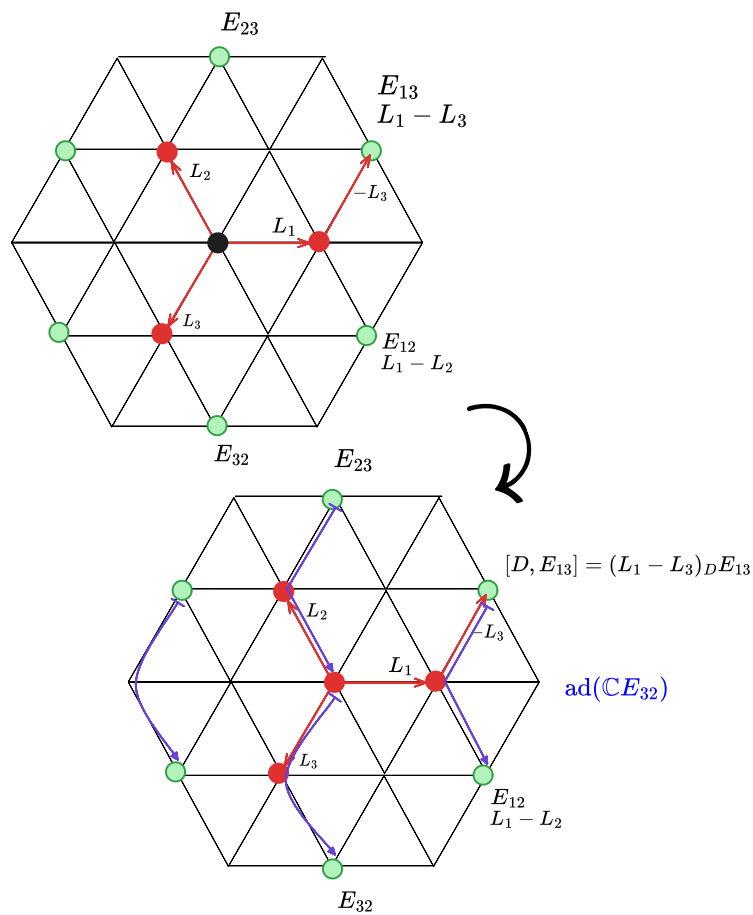
$$\left[\underbrace{H}_{\in \mathfrak{h}}, Y \right] = \text{ad}_H(Y) = \underbrace{\alpha}_{\in \mathfrak{h}^*}(H)Y$$

- But by previous calculations, the new subspaces needs to be $\mathbb{C}\{E_{ij}|i \neq j\}$ which is of $\dim = 9-3= 6$.
- This gives the decomposition we wanted because $6+2$ is the dimension of $2, \mathbb{C}$.

$$\mathfrak{h}^* \cong \frac{\mathbb{C}\{L_1, L_2, L_3\}}{\{L_1 + L_2 + L_3 = 0\}}$$

where

$$L_i() = a_i$$



$\mathfrak{sl}(2, \mathbb{C})$

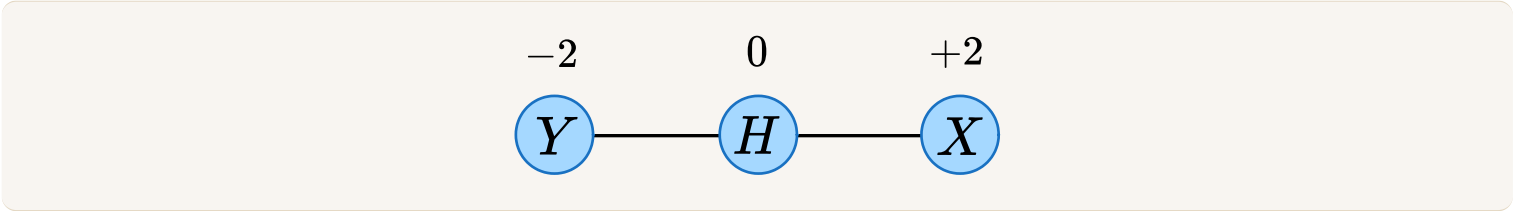
The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is the set of all **2 x 2 complex matrices with trace = 0**. This space is spanned by

$$\begin{aligned} H &:= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ (traceless diagonal)} \\ X &:= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} (= E_{12}) \\ Y &:= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} (= E_{21}) \end{aligned}$$

with

$$\begin{aligned} [H, X] &= 2X \\ [H, Y] &= -2Y \\ [X, Y] &= H \end{aligned}$$

whose structure is understood by the diagram



Thus we have

$$\mathfrak{sl}_2\mathbb{C} \cong_{\text{Alg}} \mathbb{C} \left\langle H, X, Y \left| \begin{array}{l} [H, X] = 2X \\ [H, Y] = -2Y \\ [X, Y] = H \end{array} \right. \right\rangle$$

For each $n \in \mathbb{N}$ there exists an unique $n + 1$ dimensional $\text{Vec}_{\mathbb{C}}$ irreducible representation

$$\mathfrak{sl}(2, \mathbb{C}) \rightarrow \text{EndVec}_{\mathbb{C}}(\text{Sym}^n \mathbb{C})$$

which is the n -th symmetric power of the standard representation on $\mathbb{C}^2$.

FinVec $\mathbb{C}$ irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ and $SL_{\mathbb{C}}(2)$

classification of FinVec $\mathbb{C}$ irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$

We want all representations

$$\underbrace{\mathfrak{sl}(2, \mathbb{C})}_{\cong_{\text{Alg}} \mathbb{C}\langle H, X, Y \mid [H, X] = 2X, [H, Y] = -2Y, [X, Y] = H \rangle} \rightarrow \text{EndFinVec}_{\mathbb{C}}(V)$$

up to isomorphism.

- Start with an arbitrary finite dim irrep V . How does H, X, Y act on V ?
- The adjoint action by H

$$H \overset{\text{ad}}{\curvearrowright} \mathfrak{sl}(2, \mathbb{C})$$

is diagonal. By *preservation of Jordan decomposition* under the representation

$$\mathfrak{sl}(2, \mathbb{C}) \rightarrow \mathfrak{gl}(V)$$

the action $H \curvearrowright V$ must also be diagonal: hence we have a decomposition

$$V = \bigoplus V_{\alpha}$$

where V_{α} is the subspace such that

$$v \in V_{\alpha} \implies H(v) = \alpha v$$

- If $v \in V_{\alpha}$,

$$\begin{aligned} H(X(v)) &= X(H(v)) + [H, X](v) \\ &= X(\alpha v) + 2X(v) \\ &= (\alpha + 2)X(v) \end{aligned}$$

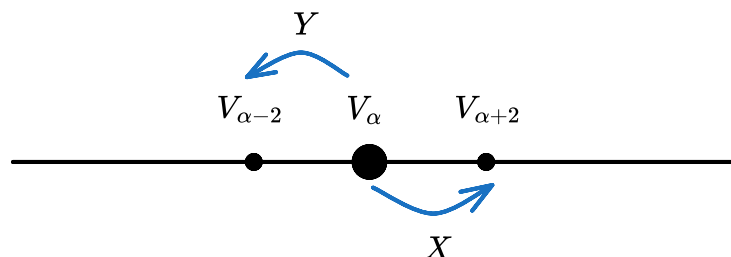
Hence, $X : V_{\alpha} \rightarrow V_{\alpha+2}$.

- If $v \in V_{\alpha}$,

$$\begin{aligned} H(Y(v)) &= Y(H(v)) + [H, Y](v) \\ &= Y(\alpha v) - 2Y(v) \\ &= (\alpha - 2)Y(v) \end{aligned}$$

Hence, $Y : V_{\alpha} \rightarrow V_{\alpha-2}$.

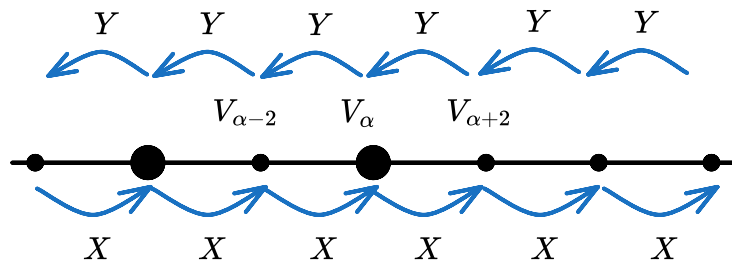
- So,



- But

$$\sum_{n \in \mathbb{Z}} V_{\alpha_0 + 2n}$$

is invariant under X, Y, H



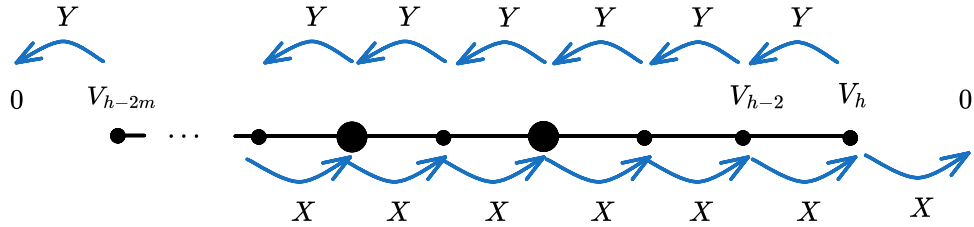
implies this subspace is invariant under the $\mathfrak{sl}_2\mathbb{C}$ action. So, irreducibility implies $V = \bigoplus V_{\alpha_0+2n}$.

- Because V is finite dim, V_{α} that appear must satisfy

$$\alpha \in \{h, h-2, \dots, h-2m\}$$

for some $h \in \mathbb{C}$ ("highest" eigenvalue for H) and $m \in \mathbb{N}$ (the "size" of the representation) where the rest of the spaces are 0.

- Thus we have



- Pick $0 \neq v \in V_n$ and then we get the subspace

$$W := \mathbb{C}\{v, Y(v), \dots, Y^m(v)\} \leq V$$

- Y preserves W by definition.
- We know $Y^k(v) \in V_{h-2k}$, so H acts by

$$H(Y^k(v)) = (h - 2k)Y^k(v)$$

- So H preserves W .
- Now $X(v) = 0$ because v belongs to the space with last V_{α} .

- Base case:**

$$\begin{aligned} X(Y(v)) &= [X, Y](v) + Y(X(v)) \\ &= H(v) + Y(0) \\ &= hv \end{aligned}$$

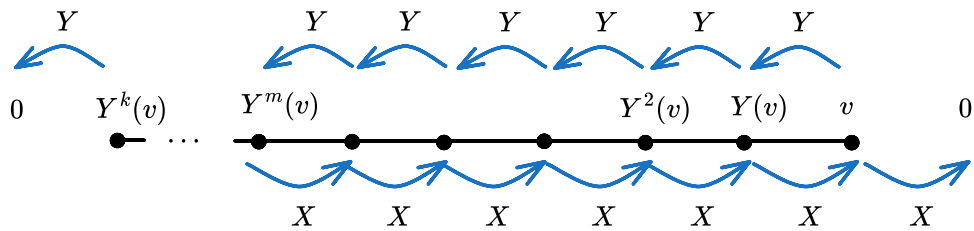
- Induction hypothesis:** We assume

$$X(Y^k(v)) = k(h - k + 1)Y^{k-1}(v)$$

- Induction step:** We obtain

$$\begin{aligned} XY^{k+1}(v) &= XY(Y^k(v)) = [X, Y](Y^k(v)) + YXY^k(v) \\ &= HY^k(v) + Y(k(h - k + 1)Y^{k-1}(v)) \\ &= (h - 2m)Y^k(v) + k(h - k + 1)Y^k(v) \\ &= h + k(h - k - 1)Y^k(v) \\ &= (k + 1)(h - k)Y^k(v) \end{aligned}$$

- Hence, $X(Y^k(v)) = k(h - k + 1)Y^{k-1}(v)$ is true for all $k \geq 1$.
- So, X preserves W .



- Hence, *by irreducibility*, we must have $V = W$. Hence, $\{v, Y(v), \dots, Y^k(v)\}$ is a basis of V .
 - $V_{n-2m} = \mathbb{C}Y^m(v)$ (one dimensional)
 - Because $Y^{m+1}(v) = 0$ we have

$$0 = XY^{m+1}(v) = (m+1)(h-m)Y^k(v)$$

but $Y^k(v) \neq 0$ by definition, so

$$h - m = 0 \implies h = m$$

and thus this implies the highest eigenvalue $h \in \mathbb{N}$ is a natural number and equals to the size of the representation.

- Hence, for any such irreducible representation, we have a $n \in \mathbb{N}$ such that the eigenvalues of H are

$$\left\{ \underbrace{n}_h, n-2, \dots, -n+2, -n \right\}$$

and dimension of the representation is $n+1$. *This does now show existence, yet.*

- Existence:** For each $n \in \mathbb{N}$ there exists a finite dim $\text{Vec}_{\mathbb{C}}$ irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$:
 - $n = 0$ is the trivial representation
 - $n = 1$ the standard representation

$$\mathfrak{sl}(2, \mathbb{C}) \rightarrow \text{EndVec}_{\mathbb{C}}(\underbrace{\mathbb{C}^2}_{\mathbb{C}\{x,y\}})$$

where $Hx = x, Hy = -y$

- and for any $n \in \mathbb{N}$ we have the representation

$$\mathfrak{sl}(2, \mathbb{C}) \rightarrow \text{EndVec}_{\mathbb{C}}(\underbrace{\text{Sym}^n \mathbb{C}^2}_{\mathbb{C}\langle x^n, x^{n-1}y, \dots, y^n \rangle})$$

such that $H(x^{n-k}y^k) = (n-2k)x^{n-k}y^k$

decomposing tensor product of irreps

geometric side

The automorphism group of $\mathbb{C}P^n$ - either as a algebraic variety or as a complex manifold - is the group >

$$PGL_{\mathbb{C}}(n+1)$$

Correspondences

For any vector space W of dim $n+1$,

$$\text{Sym}^k W^* \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{P(W)}_{= \text{space of lines in } W}\}$$

Dually,

$$\begin{aligned}\mathbf{Sym}^k W &\leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{P(W^*)}\} \\ &= \text{space of lines in } W^* \\ &= \text{space of hyperplanes in } W\end{aligned}$$

where the last correspondence is

$$\begin{aligned}\text{lines in } W^* &\leftrightarrow \text{hyperplanes in } W \\ [\phi] &\leftrightarrow \ker \phi\end{aligned}$$

Now because zero sets of polynomials are *hypersurfaces*(?)

$$P(\mathbf{Sym}^k W) \leftrightarrow \{\text{hypersurfaces of deg } k \text{ in } P(W^*)\}$$

so to derive results about $\mathbf{Sym}^k W$ we must work with $P(W^*)$.

📌 Definition. Veronese embedding

For any vector space V we have the *Veronese embedding*

$$\begin{aligned}P(V^*) &\hookrightarrow P(\mathbf{Sym}^k V^*) \\ [v] &\mapsto [v^n]\end{aligned}$$

For $\dim(V) = 2$, $P(V^*) \cong \mathbb{CP}^1$, $\mathbf{Sym}^n V^*$ is of $\dim n + 1$, so it is $P(\mathbf{Sym}^n V^*) \cong \mathbb{CP}^n$ naturally and we have

$$\begin{aligned}\iota_n : \mathbb{CP}^1 &\hookrightarrow \mathbb{CP}^n \\ x\hat{x} + y\hat{y} &\mapsto (x\hat{x} + y\hat{y})^n \\ [x, y] &\mapsto [x^n, x^{n-1}y, x^{n-2}y^2, \dots, xy^{n-1}, y^n]\end{aligned}$$

whose image is called *rational normal curve of degree n* .

Now we have the representation

$$\begin{aligned}SL_{\mathbb{C}}(2) &\curvearrowright \mathbf{Sym}^n V^* \cong \mathbb{C}^{n+1} \\ \hat{x}^k \hat{y}^{n-k} &\xrightarrow{S} (S\hat{x})^k (S\hat{y})^{n-k}\end{aligned}$$

which quotients down to

$$\begin{aligned}SL_{\mathbb{C}}(2) &\curvearrowright P\mathbf{Sym}^n V^* \cong \mathbb{CP}^n \\ \hat{x}^k \hat{y}^{n-k} &\xrightarrow{S} (S\hat{x})^k (S\hat{y})^{n-k}\end{aligned}$$

the exact action on $\mathbb{CP}^n$ that preserves the *rational normal curve*.

weight character of a $\mathfrak{sl}(2, \mathbb{C})$ representation

📌 Definition. Weight character of a $\mathfrak{sl}(2, \mathbb{C})$ representation

Because any $\text{FinVec}_{\mathbb{C}}$ representation of $\mathfrak{sl}(2, \mathbb{C})$ is reducible, we may decompose it into its irreducible pieces

$$V \cong \bigoplus_{a \in \mathbb{N}} \mathbf{Sym}^a(\mathbb{C}^2)^{\otimes d_a}$$

and further, the irreps into its weight subspaces, that is, the eigenvalue of H which is brought all together as

$$V = \sum_{n \in \mathbb{Z}} \ker(H - n\text{Id})$$

We define the **weight character** of this representation as the $\mathbb{Z}[t, t^{-1}]$ polynomial

$$\eta(V) := \sum_{n \in \mathbb{Z}} \dim \ker(H - n \text{Id}_V) t^n$$



$$\eta(V \oplus W) = \eta(V) + \eta(W)$$



$$V \cong_{\mathfrak{sl}(2, \mathbb{C})} W \iff \eta(V) = \eta(W)$$



- It is clear that $V \cong W \implies \eta(V) = \eta(W)$.
- Let $\eta(V) = \eta(W)$ then in particular their dimensions are equal because they have same weight space decomposition. Say their dimension is n .
 - The case $n = 0$ is vacuously true.
 - Consider $n > 0$ and let $\lambda \in \mathbb{Z}$ is the highest weight of both V and W . Then we may decompose

$$\begin{aligned} V &= V' + V_\lambda \\ W &= W' + W_\lambda \end{aligned}$$

where V_λ, W_λ are irr representations with highest weight λ . Now

$$\eta(V) = \eta(V') + \eta(V_\lambda) = \eta(W) = \eta(W') + \eta(W_\lambda)$$

so we have $\eta(W') = \eta(V')$.

- The *induction hypothesis* on n yields that V' and W' are isomorphic as representations.
- Thus V and W are isomorphic as representations.

We know the weight character of the irreducible reps:



$$\begin{aligned} \eta(\text{Sym}^n(\mathbb{C}^2)) &= t^{-n} + t^{-n+2} + \dots + t^{n-2} + t^n \\ &= \frac{t^{n+2} - t^{-n}}{t^2 - 1} \end{aligned}$$

But the main use of the ring of polynomials is here:



$$\eta(V \otimes W) = \eta(V)\eta(W)$$



$$\begin{aligned} H(v \otimes w) &= Hv \otimes w + v \otimes Hw \\ &= (n + m)v \otimes w \end{aligned}$$

Hence,

$$\ker_{V \otimes W}(H - k \text{Id}) = \bigoplus_{n+m=k} \ker_V(H - n \text{Id}) \otimes \ker_W(H - m \text{Id})$$

Thus

$$\begin{aligned}\eta(V \otimes W) &= \sum_{k \in \mathbb{Z}} \dim \ker_{V \otimes W}(H - k \text{Id}) t^k \\ &= \sum_{n \in \mathbb{Z}} \sum_{n+m=k} \left(\dim \ker_V(H - n \text{Id}) \right) \left(\dim \ker_V(H - m \text{Id}) \right) t^k \\ &= \left(\sum_{n \in \mathbb{Z}} \dim \ker_V(H - n \text{Id}) t^n \right) \left(\sum_{m \in \mathbb{Z}} \dim \ker_V(H - m \text{Id}) t^m \right) \\ &= \eta(V) \eta(W)\end{aligned}$$

which Laurent polynomials are achieved by representations?

$$\begin{aligned}\sum_{n \in \mathbb{N}} a_n (t^{-n} + t^{-n+2} + \dots + t^{n-2} + t^n) \\ = \sum_{i \in \mathbb{N}} \sum_j a_j (t^{-i} + t^i)\end{aligned}$$

decompositions of tensor powers of irreps into sums of irreps

• $2 \otimes 3 = 5 \oplus 3 \oplus 1$

The tensor product

$Sym^a(\mathbb{C}^2) \otimes Sym^b(\mathbb{C}^2)$	$-a$	$-a + 2$	$\dots$	$a - 2$	a
$-b$	$-a - b$	$-a - b + 2$		$a - b - 2$	$a - b$
$-b + 2$	$-a - b + 2$	$-a - b + 4$		$a - b$	$a - b + 2$
$\dots$			$\dots$		
$b - 2$	$b - a - 2$	$b - a$		$a + b - 4$	$a + b - 2$
b	$b - a$	$b - a + 2$		$a + b - 2$	$a + b$

gives the weight decomposition from top-left to bottom right

	$-a - b$	$-a - b + 2$	
	1	2	

$$Sym^a(\mathbb{C}^2) \otimes Sym^b(\mathbb{C}^2) \cong_{SL_{\mathbb{C}}(2)} \bigoplus_{i=0}^b Sym^{a+b-2i}(\mathbb{C}^2)$$

From

$$\begin{aligned}\eta(Sym^n(\mathbb{C}^2)) &= t^{-n} + t^{-n+2} + \dots + t^{n-2} + t^n \\ &= \frac{t^{n+2} - t^{-n}}{t^2 - 1}\end{aligned}$$

the weight character of the right side is

$$\begin{aligned} \sum_{i=0}^b \frac{t^{a+b-2i+2} - t^{-(a+b-2i)}}{t^2 - 1} &= \frac{\sum_{i=0}^b t^{a+b-2i+2} - \sum_{i=0}^b t^{-(a+b-2i)}}{t^2 - 1} \\ &= \frac{t^{a+2} \sum_{i=0}^b t^{b-2i} - t^{-a} \sum_{i=0}^b t^{-(b-2i)}}{t^2 - 1} \\ &= \frac{t^{a+2} - t^{-a}}{t^2 - 1} (t^n + t^{n-2} + \dots + t^{-n}) \\ &= \eta(\textit{\textbf{Sym}}^a(\mathbb{C}^2))\eta(\textit{\textbf{Sym}}^b(\mathbb{C}^2)) \end{aligned}$$

So by



$$V \cong_{\mathfrak{sl}(2,\mathbb{C})} W \iff \eta(V) = \eta(W)$$



and



$$\eta(V \otimes W) = \eta(V)\eta(W)$$



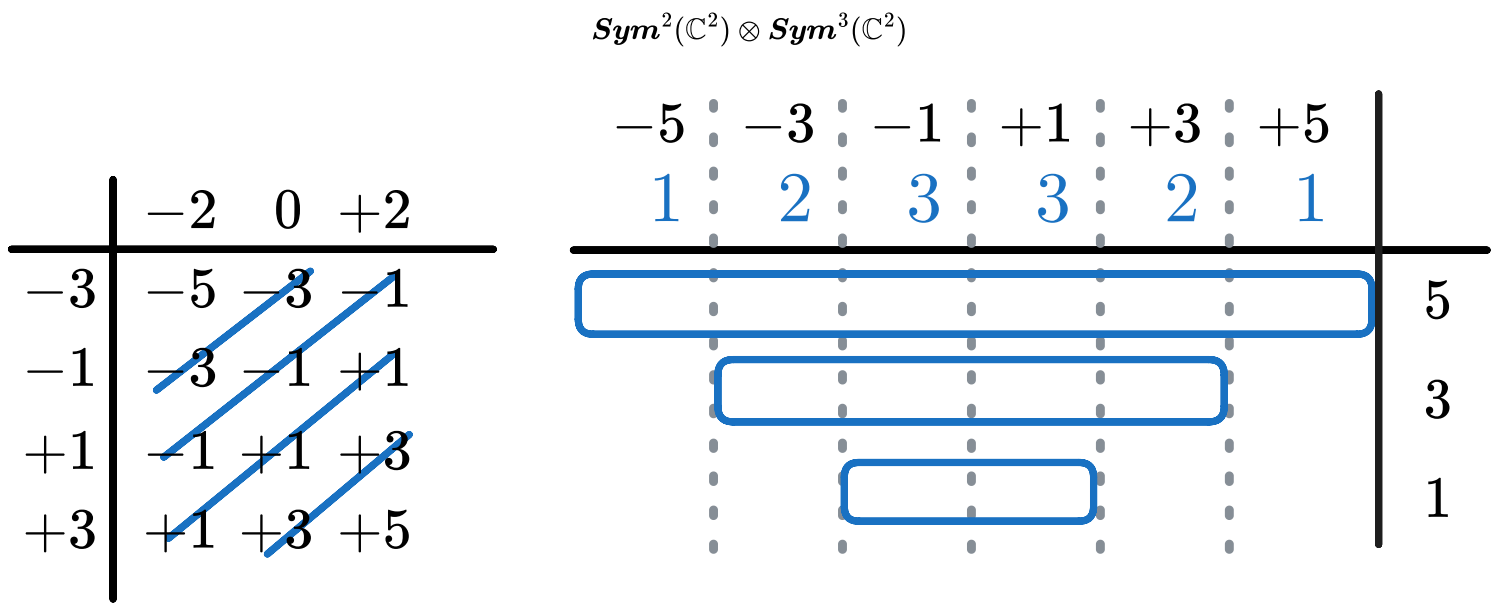
the statement follows.

decompositions of symmetric powers of irreps into sums of irreps

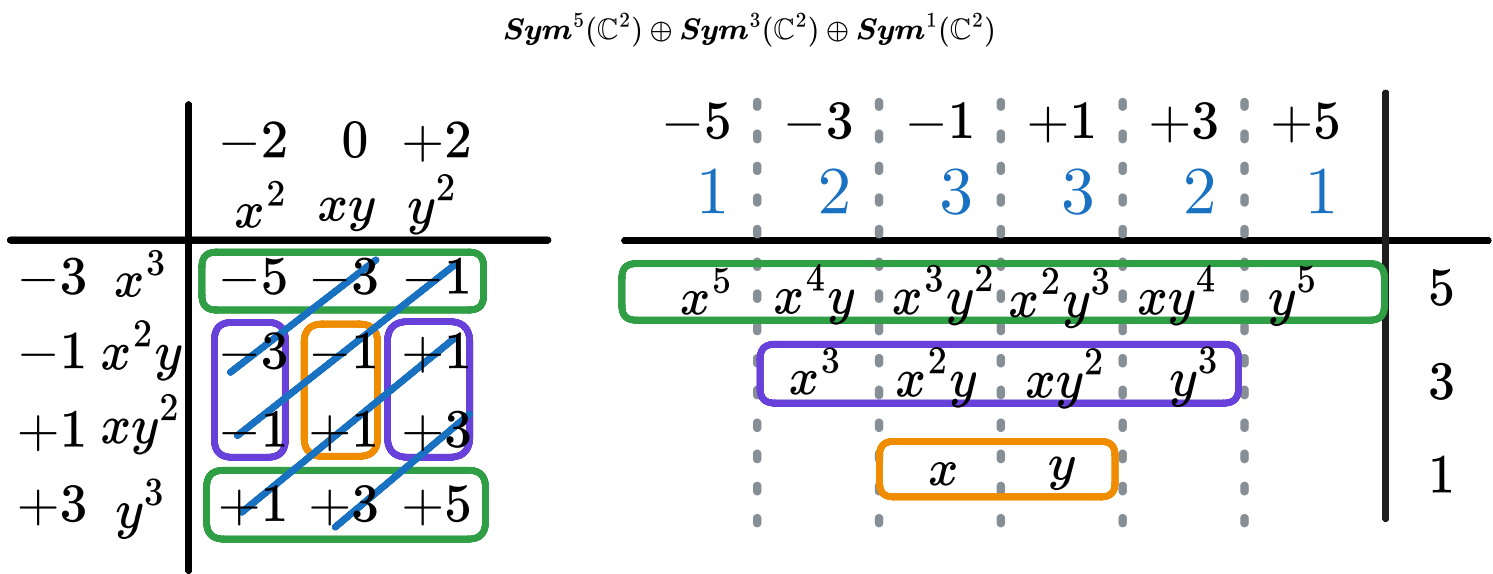
- $\textit{\textbf{Sym}}^2\textit{\textbf{Sym}}^2(\mathbb{C}^2) \cong_{SL_{\mathbb{C}}(2)} \textit{\textbf{Sym}}^0(\mathbb{C}^2) \oplus \textit{\textbf{Sym}}^4(\mathbb{C}^2)$
-

$$2 \otimes 3 \cong_{\mathfrak{sl}(2, \mathbb{C})} 5 \oplus 3 \oplus 1$$

The tensor product



decomposes as



where we choose the projection

$$\begin{array}{ccc}
 Sym^2(\mathbb{C}^2) \otimes Sym^3(\mathbb{C}^2) & \rightarrow & Sym^5(\mathbb{C}^2) \oplus Sym^3(\mathbb{C}^2) \oplus Sym^1(\mathbb{C}^2) \\
 |x^2, x^3\rangle & \mapsto & |x^5\rangle
 \end{array}$$

- the weight character

$$\begin{aligned}
 & (t^{-2} + 1 + t^2)(t^{-3} + t^{-1} + t + t^3) \\
 &= t^{-5} + 2t^{-3} + 3t^{-1} + 3t + 2t^3 + t^5
 \end{aligned}$$

- the quotient projection

$$Sym^2(\mathbb{C}^2) \otimes Sym^3(\mathbb{C}^2) \rightarrow Sym^5(\mathbb{C}^2)$$

is multiplication of polynomials but carefully

•

$$\mathbf{Sym}^2(2) \cong_{\mathfrak{sl}(2, \mathbb{C})} 0 \oplus 4$$

$$\mathbf{Sym}^2 \mathbf{Sym}^2(\mathbb{C}^2) \cong_{\mathfrak{sl}(2, \mathbb{C})} \mathbf{Sym}^0(\mathbb{C}^2) \oplus \mathbf{Sym}^4(\mathbb{C}^2)$$

using weights

Handwritten notes on a chalkboard illustrating the decomposition of $\mathbf{Sym}^2 \mathbf{Sym}^2 V$ into $\mathbf{Sym}^0 V$ and $\mathbf{Sym}^4 V$ using weights. The notes show the action of the Cartan subalgebra H on the weight spaces of V and $\mathbf{Sym}^2 V$. The weight spaces of V are labeled x (weight +1) and y (weight -1). The weight spaces of $\mathbf{Sym}^2 V$ are labeled z_0 (weight -2), z_1 (weight 0), and z_2 (weight +2). The decomposition is shown as $\mathbf{Sym}^2 \mathbf{Sym}^2 V \rightarrow \mathbf{Sym}^0 V \oplus \mathbf{Sym}^4 V$. The $\mathbf{Sym}^0 V$ component is a single box labeled 1 (weight 0). The $\mathbf{Sym}^4 V$ component is a stack of five boxes with weights -3, -1, 0, +1, and +3, labeled $x^4, x^3y, x^2y^2, xy^3, y^4$ respectively.

		$Sym^2(\mathbb{C}^2)$										
		-2	0	+2								
		Z_0	Z_1	Z_2								
$Sym^2(\mathbb{C}^2)$	-2	-4										
	Z_0	Z_0^2										
	0	-2	0									
	Z_1	$Z_0 Z_1$										
$Sym^2 Sym^2(\mathbb{C}^2)$	+2	0	+2	+4								
	Z_2	$Z_0 Z_2$										

decomposition as homogeneous polynomials

Correspondences

For any vector space W of dim $n + 1$,

$$\mathbf{Sym}^k W^* \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{P(W)}\} \\ = \text{space of lines in } W$$

Dually,

$$\begin{aligned}\mathbf{Sym}^k W &\leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{P(W^*)}\} \\ &= \text{space of lines in } W^* \\ &= \text{space of hyperplanes in } W\end{aligned}$$

where the last correspondence is

$$\begin{aligned}\text{lines in } W^* &\leftrightarrow \text{hyperplanes in } W \\ [\phi] &\leftrightarrow \ker \phi\end{aligned}$$

Now because zero sets of polynomials are *hypersurfaces*(?)

$$P(\mathbf{Sym}^k W) \leftrightarrow \{\text{hypersurfaces of deg } k \text{ in } P(W^*)\}$$

so to derive results about $\mathbf{Sym}^k W$ we must work with $P(W^*)$.

We understand

- $\mathbf{Sym}^2 \mathbf{Sym}^2 V = \{\text{homg deg 2 polynomials on } P(\mathbf{Sym}^2 V)^*\}$

- for $V = \mathbb{C}^2 = \mathbb{C}\{x, y\}$, we have

$$\begin{aligned}\mathbf{Sym}^2 \mathbb{C}\{x, y\} &\cong \mathbb{C}\{x^2, xy, y^2\} \cong \mathbb{C}\{Z_0, Z_1, Z_2\} \\ x^2 &\mapsto Z_0 \\ xy &\mapsto Z_1 \\ y^2 &\mapsto Z_2\end{aligned}$$

- which gives

$$\mathbf{Sym}^2 \mathbf{Sym}^2 \mathbb{C}^2 = \underbrace{\mathbb{C}\left\{ \begin{array}{c} Z_0^2, Z_1^2, Z_2^2, \\ Z_0 Z_1, Z_1 Z_2, Z_2 Z_0 \end{array} \right\}}_{\text{homg polynomials on } P\mathbb{C}\{Z_0, Z_1, Z_2\} \cong \mathbb{C}P^2}$$

- thus using

- for $V = \mathbb{C}^2 = \mathbb{C}\{x, y\}$, we have

$$\begin{aligned}\mathbf{Sym}^2 \mathbb{C}\{x, y\} &\cong \mathbb{C}\{x^2, xy, y^2\} \cong \mathbb{C}\{Z_0, Z_1, Z_2\} \\ x^2 &\mapsto Z_0 \\ xy &\mapsto Z_1 \\ y^2 &\mapsto Z_2\end{aligned}$$

we have the evaluation map

$$\begin{aligned}\underbrace{\mathbf{Sym}^2 \mathbf{Sym}^2 \mathbb{C}^2} &\rightarrow \underbrace{\mathbf{Sym}^4 \mathbb{C}^2} \\ \mathbb{C}\left\{ \begin{array}{c} Z_0^2, Z_1^2, Z_2^2, \\ Z_0 Z_1, Z_1 Z_2, Z_2 Z_0 \end{array} \right\} &\rightarrow \mathbb{C}\left\{ \begin{array}{c} x^4 \\ x^3 y, x^2 y^2, xy^3 \\ y^4 \end{array} \right\} \\ Z_0^2 &\mapsto x^4 \\ Z_1^2 &\mapsto x^2 y^2 \\ Z_2^2 &\mapsto y^4 \\ Z_0 Z_1 &\mapsto x^3 y \\ Z_1 Z_2 &\mapsto xy^3 \\ Z_2 Z_0 &\mapsto x^2 y^2\end{aligned}$$

- this map surjective because the chosen basis is contained in the image
- and it is easily seen

$$Z_2 Z_0 - Z_1^2 \mapsto 0$$

and this polynomial spans the kernel because the map is surjective and it goes from a dim 6 to a dim 5 space

- So we get an **exact sequence**

$$0 \rightarrow \mathbb{C}\{Z_2 Z_0 - Z_1^2\} \hookrightarrow \mathbf{Sym}^2 \mathbf{Sym}^2 \mathbb{C}^2 \rightarrow \mathbf{Sym}^4 \mathbb{C}^2 \rightarrow 0$$

- But what about the split $\mathbf{Sym}^4 \mathbb{C}^2 \hookrightarrow \mathbf{Sym}^2 \mathbf{Sym}^2 \mathbb{C}^2$? We must define something like

$$\begin{aligned} \mathbf{Sym}^4 \mathbb{C}^2 &\hookrightarrow \mathbf{Sym}^2 \mathbf{Sym}^2 \mathbb{C}^2 \\ x^4 &\mapsto Z_0^2 \\ x^3 y &\mapsto Z_0 Z_1 \\ x^2 y^2 &\mapsto \frac{1}{2}(Z_1^2 + Z_0 Z_2) \\ x y^3 &\mapsto Z_1 Z_2 \\ y^4 &\mapsto Z_2^2 \end{aligned}$$

$\mathbb{C}P^2$ geometry behind the decomposition

For $n = 2$ we have the embedding

$$[x, y] \mapsto [x^2, xy, y^2]$$

whose image satisfies the equation

$$F(Z_0, Z_1, Z_2) := Z_0 Z_2 - (Z_1)^2 = 0$$

in $\mathbb{C}P^2$ with coordinates

$$[Z_0, Z_1, Z_2] := [x^2, xy, y^2]$$

whose zero set is called the *plane conic* (rational normal curve of degree 2).

The action

$$\begin{aligned} SL_{\mathbb{C}}(2) &\curvearrowright P\mathbf{Sym}^2 V^* \cong \mathbb{C}P^2 \\ \hat{x}^k \hat{y}^{n-k} &\xrightarrow{S} (S\hat{x})^k (S\hat{y})^{n-k} \end{aligned}$$

is exactly the group of motions of $\mathbb{C}P^2$ that carry the plane conic c_2 onto itself

$$\begin{array}{ccc} P(V) & \xleftarrow{i_2} & P(\mathbf{Sym}^2 V) \\ \downarrow \scriptstyle SL_{\mathbb{C}}(2) & \begin{array}{c} [v] \mapsto [v^2] \\ \downarrow \scriptstyle s \\ [sv] \mapsto [sv^2] \end{array} & \downarrow \\ P(V) & \xleftarrow{i_2} & P(\mathbf{Sym}^2 V) \end{array}$$

Thus the action

$$SL_{\mathbb{C}}(2) \curvearrowright \mathbf{Sym}^2(\mathbf{Sym}^2(V)) \leftrightarrow \{\text{homog polynomials of deg } k \text{ on } \underbrace{P(W^*)}_{\mathbf{Sym}^2(V)}\}$$

must preserve $\mathbb{C}\{F\}$ spanned the polynomial for c_2 .

The split

$$\boldsymbol{Sym}^4(V) \rightarrow$$

Hence

$$0 \rightarrow \underbrace{\mathbb{C}\{F\}}_{\boldsymbol{Sym}^0(V)} \hookrightarrow \boldsymbol{Sym}^2(\boldsymbol{Sym}^2(V)) \rightarrow \boldsymbol{Sym}^4(V) \rightarrow 0$$

is exact as $SL_{\mathbb{C}}(2)$ -modules, that splits giving us an isomorphism

$$\boldsymbol{Sym}^2(\boldsymbol{Sym}^2(V)) \cong_{SL_{\mathbb{C}}(2)} \boldsymbol{Sym}^4(V) \oplus \boldsymbol{Sym}^0(V)$$

Compact Lie groups and their representations (2022)- Prof. M. S. Raghunathan, Centre for Basic Sciences, Mumbai

lecture 1

(Topological) compact, 2-countable groups vs compact Lie groups

- **Harr measure**: a left invariant Borel finite measure μ
- any representation has a invariant inner product
 - start with any inner product $(\cdot, \cdot)$ and define

$$\langle u, v \rangle = \int_{(G, \mu)} \langle \rho(g)u, \rho(g)w \rangle$$

•

Ad* orbits are symplectic manifolds

Definition. Symplectic structure on coadjoint orbits of Lie groups

Let

$$\Omega \subset \mathfrak{g}^*$$

be a coadjoint orbit $\text{Ad}^* : G \curvearrowright \mathfrak{g}^*$ of a Lie group G . Let $F \in \Omega$.

- By definition Ω is a transitive (left) G -space with stabilizer of F being $\text{Ad}_{G,F}^*$ (a closed subgroup).
- Thus the surjective smooth map

$$\begin{aligned} G &\rightarrow \Omega \\ g &\mapsto \text{Ad}_g^*(F) \end{aligned}$$

with descends to bijection

$$\frac{G}{\text{Ad}_{G,F}^*} \rightarrow \Omega$$

giving a smooth structure on the orbit Ω making it a *smooth immersed manifold* in $\mathfrak{g}^*$.

We construct a anti-symmetric bilinear map

$$\begin{aligned} \omega_F : \frac{\mathfrak{g}}{\text{Lie}(\text{Ad}_{G,F}^*)} \times \frac{\mathfrak{g}}{\text{Lie}(\text{Ad}_{G,F}^*)} &\rightarrow \mathbb{R} \\ (X, Y) &\mapsto F([X, Y]) \end{aligned}$$

Proposition:

- The bilinear map

$$\begin{aligned} \omega_F : \frac{\mathfrak{g}}{\text{Lie}(\text{stab}_G(F))} \times \frac{\mathfrak{g}}{\text{Lie}(\text{stab}_G(F))} &\rightarrow \mathbb{R} \\ (X, Y) &\mapsto F([X, Y]) \end{aligned}$$

defines a Ad^* -invariant **symplectic form** on the orbit $\Omega \subset \mathfrak{g}^*$.

- The coadjoint action $G \curvearrowright \Omega$ is **Hamiltonian** with moment map

$$\iota : \Omega \hookrightarrow \mathfrak{g}^*$$

Thus

$$\text{Ad}^* : G \curvearrowright (\Omega, \omega) \subset \mathfrak{g}^*$$

is a **homogeneous Hamiltonian G -manifold**.

[1]

[2]

Lie group	coadjoint orbits
$SO(3, \mathbb{R}), SU(2, \mathbb{C})$	$\{0\}, S^2$
compact	Kahler [3]
nilpotent	?

1. page 48 of the pdf

- [Matthias Peter - The Orbit Method for Compact Connected Lie Groups](#)



2. studenttheses.uu.nl/bitstream/handle/20.500.12932/7207/Maes%2CJMA2011.pdf?sequence=1&isAllowed=y#page=40.00 ↩

3. prop 3.1.21 of the pdf

- [Matthias Peter - The Orbit Method for Compact Connected Lie Groups](#)



IMSC-Linear algebraic groups

...the first 4 lectures were missed, I suggest reading from the start of chapter 2, up to Proposition 2.2.5 of Springer's textbook 'Linear algebraic groups'. This is where he starts from (after an overview).

GSG on Anosov representations

✦ Abstract

Everyone is invited to the next talk of the Graduate Students' Group seminar, whose details are as follows:

Speaker: [Tathagata Nayak](#)

Title:- On Character Variety of Anosov Representations

Abstract: Let Γ be the fundamental group of an k -punctured, $k \geq 0$, closed connected orientable surface of genus $g \geq 2$. In this talk, it will be shown that the character variety of the $(Q+, Q-)$ -Anosov irreducible representations, resp. the character variety of the $(P+, P-)$ -Anosov Zariski dense representations of Γ into $SL(n, \mathbb{C})$, $n \geq 2$, is a complex manifold of complex dimension $(2g + k - 2)(n^2 - 1)$. For $\Gamma = \pi_1(\Sigma_g)$, these character varieties are holomorphic symplectic manifolds. This talk is based on a joint work with Prof. Krishnendu Gongopadhyay.

Date: 25 Jan 2025 (Saturday)

Time: 4 30 PM - 5 30 PM

Venue: AB2 2A/2B

We look forward to your presence. Feel free to ask questions to the speaker between and after the talk.

<https://arxiv.org/pdf/1108.0733>

<https://arxiv.org/pdf/2409.07316>

Gauss map

[1]

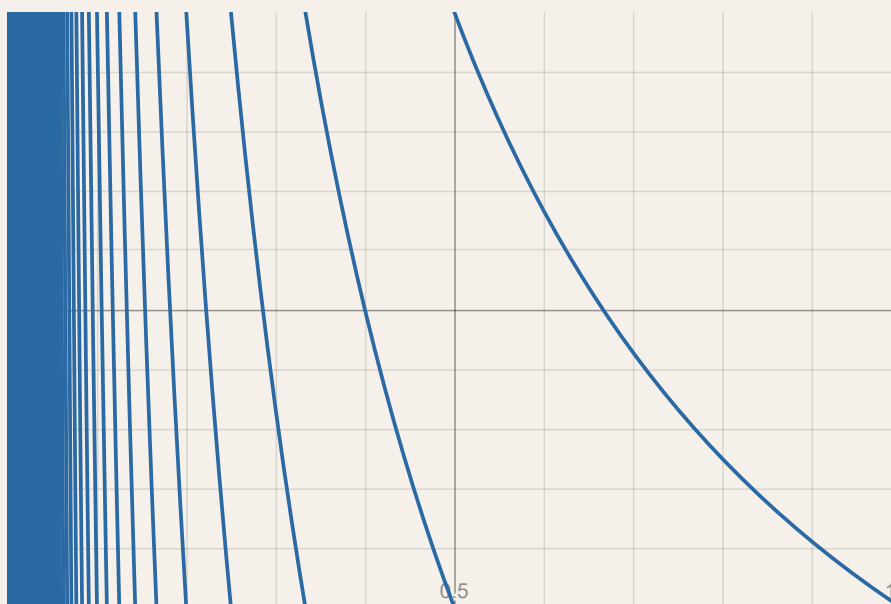
[2]

Definition. Gauss map

The Gauss map

$$G : [0, 1] \rightarrow [0, 1]$$

$$x \mapsto \begin{cases} 0 & x = 0 \\ \left\{ \frac{1}{x} \right\} = \frac{1}{x} \bmod 1 & x \neq 0 \end{cases}$$



• As

$$\left\lceil \frac{1}{x} \right\rceil = n \iff n \leq \frac{1}{x} < n+1 \iff \frac{1}{n+1} < x \leq \frac{1}{n}$$

• so

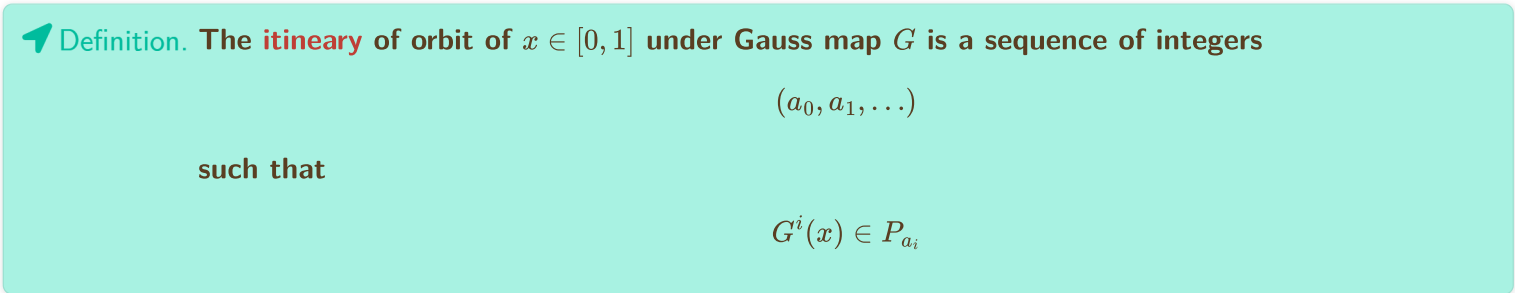
$$\frac{1}{n+1} < x < \frac{1}{n} \iff x \in \underbrace{\left(\frac{1}{n+1}, \frac{1}{n} \right]}_{P_n} \implies G(x) = \frac{1}{x} - n$$

• Thus G restricted to an interval

$$G : P_n := \left(\frac{1}{n+1}, \frac{1}{n} \right] \rightarrow [0, 1)$$

$$x \mapsto \frac{1}{x} - n$$

a *branch* of G , is monotonically decreasing, $\mathbb{R}$ -analytic in the interior, and surjective onto $[0, 1)$.



- $$\left\lfloor \frac{1}{x} \right\rfloor = n \iff n \leq \frac{1}{x} < n+1 \iff \frac{1}{n+1} < x \leq \frac{1}{n}$$

$$\frac{1}{n+1} < x < \frac{1}{n} \iff x \in \underbrace{\left(\frac{1}{n+1}, \frac{1}{n} \right]}_{P_n} \implies G(x) = \frac{1}{x} - n$$
$$x \in P_n \iff \left\lfloor \frac{1}{x} \right\rfloor = n$$

- $$G^0(x) = x \in P_{a_0} \implies a_0 = \left\lceil \frac{1}{x} \right\rceil$$

- **Base case:**

$$G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor = \frac{1}{x} - a_0 \iff x = \frac{1}{a_0 + G(x)}$$

- **Induction hypothesis:** We shall assume

$$a_n = \left\lfloor \frac{1}{G^n(x)} \right\rfloor$$

and

$$x = \frac{1}{a_0 + \frac{1}{a_1 + \dots \frac{1}{a_n + G^{n+1}(x)}}}$$

[3]

- **Induction step:** We know

$$G^{n+1}(x) \in P_{a_{n+1}} \implies a_{n+1} = \left\lfloor \frac{1}{G^{n+1}(x)} \right\rfloor$$

Then

$$G^{n+2}(x) = \frac{1}{G^{n+1}(x)} - \underbrace{\left\lfloor \frac{1}{G^{n+1}(x)} \right\rfloor}_{a_{n+1}} \iff G^{n+1}(x) = \frac{1}{a_{n+1} + G^{n+2}(x)}$$

This gives us

$$x = \frac{1}{a_0 + \frac{1}{a_1 + \dots \frac{1}{a_n + \frac{1}{a_{n+1} + G^{n+2}(x)}}}}$$

Thus



Definition. The **itinerary** of orbit of $x \in [0, 1]$ under Gauss map G is a sequence of integers

$$(a_0, a_1, \dots)$$

such that

$$G^i(x) \in P_{a_i}$$

Then

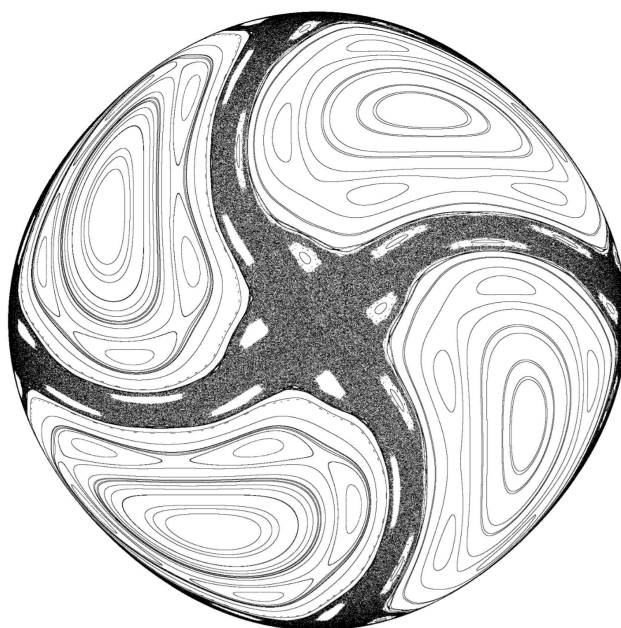
$$x = \frac{1}{a_0 + \frac{1}{a_1 + \dots \frac{1}{a_n + \frac{1}{a_{n+1} + G^{n+2}(x)}}}}$$

3. **Definition. Continued fraction notation**

$$b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \dots}}} \equiv b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \dots$$



McMullen - Ergodic theory, geometry and dynamics



✧ Abstract

Math 275 - Tu Th 12-1:15 - Virtual Reality
Harvard University - Fall 2020

● **Instructor:** [Curtis T McMullen](#)

● **Description:** A survey of fundamental results and current research. Topics may include:

- Dynamics on the circle and the torus
- Lie groups and ergodic theory
- Hyperbolic surfaces and $SL_2(\mathbb{R})$
- Kazhdan's property T and $SL_3(\mathbb{R})$
- Amenability and expanding graphs
- Martingales and Furstenberg's theorem
- Hyperbolic 3-manifolds and Mostow rigidity
- Ratner's theorem
- Conjectures of Oppenheim and Littlewood
- Planes in hyperbolic 3-manifolds
- Dynamics on moduli space M_g

● Main Course Notes

- C. McMullen, [Notes on Ergodic Theory](#)
These notes will evolve during the course; be sure to use the current version, linked above.
- *See also:* C. McMullen, [Notes on Teichmüller Theory and Complex Dynamics](#)

● Suggested Texts

- M. B. Bekka and M. Mayer, [Ergodic theory and topological dynamics of group actions on homogeneous spaces](#), Cambridge University Press, 2000.
- Bekka, de la Harpe and Valette, [Kazhdan's Property \(T\)](#), 2007.

- Benedetti and Petronio, [Lectures on Hyperbolic Geometry](#), Springer-Verlag, 1992.
- E. Ghys, [Dynamique des flots unipotents sur les espaces homogènes](#), Sem. Bourbaki 1991/92; Asterisque 206.
- M. Gromov, [Volume and bounded cohomology](#)
- R. Mañé, [Ergodic Theory and Differentiable Dynamics](#)
- D. Witte Morris, [Ratner's Theorems on Unipotent Flows](#), Chicago Lectures in Math. Series, 2005.
- J. Ratcliffe, [Foundations of Hyperbolic Manifolds](#), 2nd Edition. Springer, 2006.
- W. P. Thurston, [Three-Dimensional Geometry and Topology](#), Princeton University Press, 1997.

- people.math.harvard.edu/~ctm/home/text/class/harvard/275/20/html/index.html
 - notes <https://people.math.harvard.edu/~ctm/home/text/class/harvard/275/20/html/home/course/course.pdf>
- [McMullen - Ergodic theory, geometry and dynamics \(2020\).pdf](#)

powers of 2 that starts with 1 in decimal

Here log has base 10.

We want to know when we have

$$10^n \leq 2^k < 2(10^n)$$

taking log this becomes

$$n \leq k \log_{10} 2 < (\log_{10} 2) + n$$

modulo $\mathbb{Z}$ it boils down to how often

$$\begin{aligned} 0 &\leq k \log_{10} 2 \bmod \mathbb{Z} \leq \log_{10} 2 \\ \iff k \log_{10} 2 \bmod \mathbb{Z} &\in [0, \log_{10} 2] \subset [0, 1] \end{aligned}$$

dynamics on compact metrizable spaces

- $\mathcal{C}(X, \mathbb{R})$ is a Banach space
- $\mathcal{C}(X, \mathbb{R})^*$ is the space of Borel $\mathbb{R}$ -measures (finite, signed)
- $P(X)$ is compact in weak* topology by

 **(Banach–Alaoglu theorem)** Let X be a normed space and

$$B := \{T \in X^* \mid \|T\| \leq 1\}$$

Then this ball is compact in the weak-* topology on X^* .

- $$X \xrightarrow{T} X$$

gives pullback

$$\mathcal{C}(X) \xleftarrow{T^*} \mathcal{C}(X)$$

which gives pushforward of measures

$$M(X) \xrightarrow{T_*} M(X)$$

- $P(X)^T$ are T -invariant probability measures

 Any continuous map $T : X \rightarrow X$ on a non-empty compact metric space admits an invariant measure.

Proposition: T is μ -ergodic $\iff \mu$ is an extremal point of $P(X)^T$

💡 Let T is not μ -ergodic, then $X = A \cup A^c$ then

$$\mu = \mu(A)\mu_A + \mu(B)\mu_B$$

•

Einsiedler and Ward - Ergodic Theory

[#book/series](#)

- [Einsiedler and Ward - Ergodic Theory \(2011\).pdf](#)
- [Entropy | Books](#)
- [Homogeneous | Books](#)

Uri Bader - A course in linear groups and ergodic theory

$$SL(2, \mathbb{R}) \curvearrowright \mathbb{R}^2$$

Consider

$$\underbrace{SL(2, \mathbb{R})}_{=: G} \curvearrowright \underbrace{\mathbb{R}^2}_{=: X}$$

There are two orbits

$$\{0\}$$

and

$$\mathbb{R}^2 \setminus \{0\}$$

(disjoint of course).

By orbit-stabilizer theorem

$$\mathbb{R}^2 \setminus \{0\} \cong \frac{G}{\mathcal{U}}$$

where stabilizer is the subgroup

$$\mathcal{U} := \text{stab}_G \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left\{ \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \right\} = \exp \left\{ \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} \right\} \cong (\mathbb{R}, +)$$

$SL(2, \mathbb{R})$ -maps

? Question

Find all G -invariant maps

$$\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

that is

$$\forall g \in G, x \in \mathbb{R}^2, \phi(gx) = g\phi(x)$$

Because G -orbits will go to G -orbits under such a map we conclude

$$\phi(0) = 0$$

But if we choose where one point of a orbit will go, then we know where the other elements will go

$$\underbrace{g\phi(x)}_{\text{known}} = \underbrace{\phi(gx)}_{\text{unkown}}$$

But to know

$$\phi \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

we observe

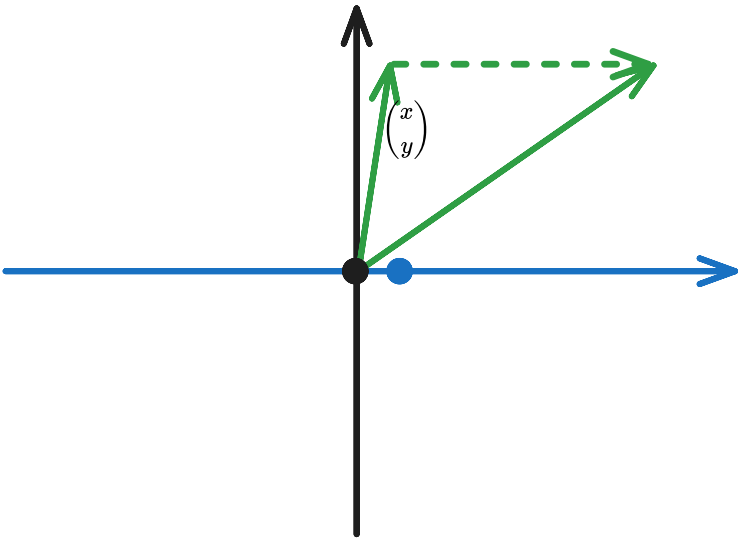
$$g \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \phi \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \phi(g \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = g\phi \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

means

$$\mathcal{U} \subset \text{stab}_G(\phi\begin{pmatrix} 1 \\ 0 \end{pmatrix})$$

So we must look at *the orbit structure of $\mathcal{U} \curvearrowright \mathbb{R}^2$* to know what point does it *stabilizes*:

$$\begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \in \mathcal{U} \implies \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \alpha \begin{pmatrix} y \\ 0 \end{pmatrix}$$



✓ [flow](#) ✓

The dynamics is "tame".

So

$$\mathcal{U} \subset \text{stab}_G(\phi\begin{pmatrix} 1 \\ 0 \end{pmatrix}) \iff \phi\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

Then

$$\phi(y) = \phi(g\begin{pmatrix} 1 \\ 0 \end{pmatrix}) = g\phi\begin{pmatrix} 1 \\ 0 \end{pmatrix} = g\begin{pmatrix} \alpha \\ 0 \end{pmatrix} = \alpha g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \alpha y$$

Thus, every G -invariant map $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a scaling

$$\phi(x) = \alpha x$$

by some $\alpha \in \mathbb{R}$, which is invertible if $\alpha \in \mathbb{R}^\times$.

✎ Exercise

$$\text{Aut}_G \text{Set} \left(\frac{G}{H} \right) \cong \frac{N_G(H)}{H}$$

For our case $H = \mathcal{U}$

$$N_G(\mathcal{U}) = \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\} \cong \mathbb{R}^\times \times \mathbb{R}$$

and

$$\frac{N_G(\mathcal{U})}{\mathcal{U}} \cong \mathbb{R}^\times$$

$SL(2, \mathbb{Z})$ -maps

We may ask


Question

Find all Borel maps

$$\mathbb{R}^2 \rightarrow \mathbb{R}^2$$

that commutes with $\Gamma := SL(2, \mathbb{Z})$ defined upto Lebesgue measure-zero sets.

whose answer is


Every Borel measurable $SL(2, \mathbb{Z})$ -map $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is essentially a scaling upto Lebesgue measure-zero sets.

But what is special about Γ ? The fact that it is a "*lattice*" in G that is:

Proposition: $\Gamma < G$ is discrete and the space

$$G \curvearrowright \frac{G}{\Gamma}$$

has a G -invariant probability measure.

The subgroup

$$A := \left\{ \begin{bmatrix} \lambda & \\ & \lambda^{-1} \end{bmatrix} \mid \lambda \in \mathbb{R}^\times \right\} < G = SL(2, \mathbb{R})$$

which acts by $\begin{bmatrix} -1 & \\ & -1 \end{bmatrix}$ along with the flow

$$\left\{ \exp \left(\alpha \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \right) \mid \alpha \in \mathbb{R} \right\}$$


 flow of A



This dynamics is "*tame*" as in the orbits are "locally closed".

"Tame dynamics"	"Wild dynamics"
action by $\mathcal{U}, A$ above on $\mathbb{R}^2$	irrational torus rotations
	$\Gamma \curvearrowright \frac{G}{\mathcal{U}}$
	$\mathcal{U} \curvearrowright \frac{G}{\Gamma}$

There are no $SL(2, \mathbb{R})$ -invariant metric on $\mathbb{R}^2$

Every continuous

$$d : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow [0, \infty)$$

$$d(x, y) = d(y, x)$$

$$d(x, x) = 0$$

satisfying triangle inequality which is $SL(2, \mathbb{R})$ -invariant, that is,

$$g \in SL(2, \mathbb{R}) \implies d(gx, gy) = d(x, y)$$

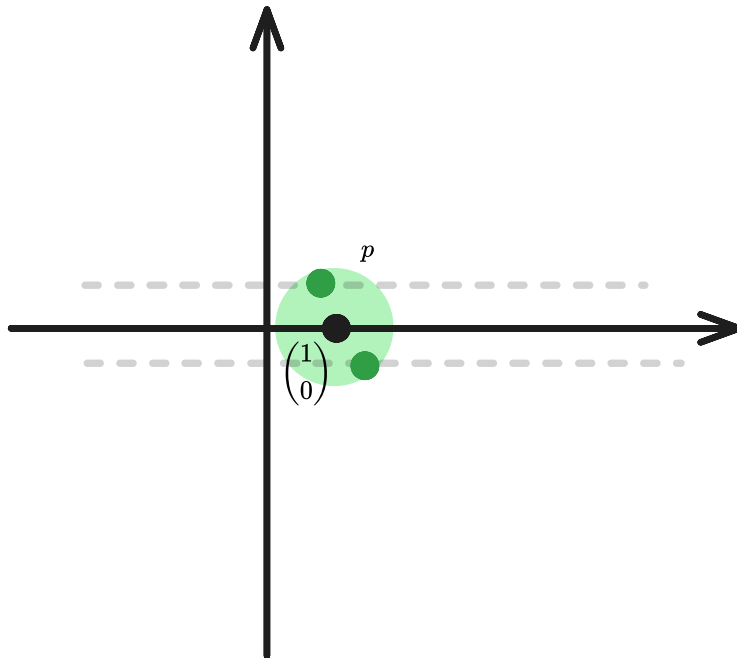
is trivial, as in $d \equiv 0$.

☀ Fix $\epsilon > 0$.

- The restriction $d\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, -\right) : \mathbb{R}^2 \rightarrow [0, \infty)$ is continuous so preimage of $(0, \epsilon)$ contains a standard ball say B_ϵ .
- Because $\mathcal{U}$ fixes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

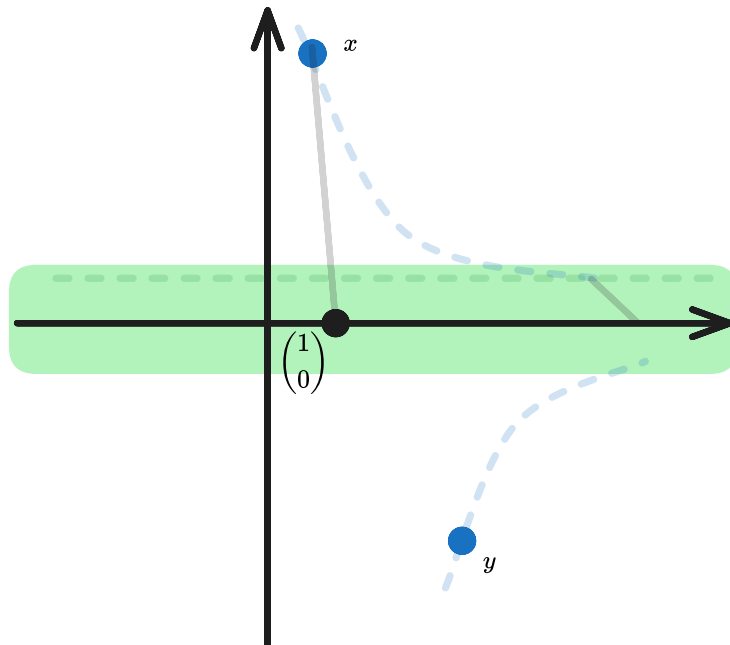
$$d(gp, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = d(p, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) < \epsilon$$

for any $p \in B_\epsilon$.



-
- Thus the $d\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, -\right)$ -preimage of $(0, \epsilon)$ actually contains a horizontal strip $U_\epsilon := \mathbb{R} \times (-\epsilon, \epsilon)$.
- Now pick $x, y \in \mathbb{R}^2$ and flow them along A so that they both reach the U_ϵ , so

$$gx \in U_\epsilon$$



-
- By triangle-inequality

$$d(x, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = d(gx, g \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \leq d(gx, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) + d(g \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \leq 2\epsilon$$

- By triangle-inequality

$$d(x, y) \leq d(x, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) + d(y, \begin{pmatrix} 1 \\ 0 \end{pmatrix}) = 4\epsilon$$

- So

$$d(x, y) < 4\epsilon$$

for all $\epsilon > 0$. Thus $d \equiv 0$.

Action of $\mathcal{U}$ on G/Γ is ergodic

Corollary of

Every continuous

$$d : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow [0, \infty)$$

$$d(x, y) = d(y, x)$$

$$d(x, x) = 0$$

satisfying triangle inequality which is $SL(2, \mathbb{R})$ -invariant, that is,

$$g \in SL(2, \mathbb{R}) \implies d(gx, gy) = d(x, y)$$

is trivial, as in $d \equiv 0$.

: The action of $\mathcal{U}$ on G/Γ is ergodic

Definition. Ergodic actions

An action $G \curvearrowright X$ with an invariant measure class is **ergodic** if either of these is true

- $\forall A \subseteq X$ measurable and invariant A is either a null set or a full measure set
- $L^\infty(X)^G$ is one dimensional

- (if X has a G -invariant probability measure) $L^2(X)^G$ is one dimensional

We observe the action of G on G/Γ is ergodic because the action is transitive?



Let X is a G -space, $\mathcal{U} < G$ and $\Gamma < G$ is a lattice. Then we have a G -invariant measure on G/Γ so we may consider the L^2 space on it. Then

$$L^2(G/\Gamma)^{\mathcal{U}} \cong \text{Hom}_G \left(\frac{G}{\mathcal{U}}, L^2(G/\Gamma) \right)$$

Every continuous

$$\begin{aligned} d : \mathbb{R}^2 \times \mathbb{R}^2 &\rightarrow [0, \infty) \\ d(x, y) &= d(y, x) \\ d(x, x) &= 0 \end{aligned}$$

satisfying triangle inequality which is $SL(2, \mathbb{R})$ -invariant, that is,

$$g \in SL(2, \mathbb{R}) \implies d(gx, gy) = d(x, y)$$

is trivial, as in $d \equiv 0$.

implying

$$\text{Hom}_G \left(\frac{G}{\mathcal{U}}, L^2(G/\Gamma) \right) \cong L^2(G/\Gamma)^G \cong \mathbb{C}$$

by

We observe the action of G on G/Γ is ergodic because the action is transitive?

Ergodic group actions

Definition. Ergodic actions

An action $G \curvearrowright X$ with an invariant measure class is **ergodic** if either of these is true

- $\forall A \subseteq X$ measurable and invariant A is either a null set or a full measure set
- $L^\infty(X)^G$ is one dimensional
- (if X has a G -invariant probability measure) $L^2(X)^G$ is one dimensional

Let X is a G -space, $\mathcal{U} < G$ and $\Gamma < G$ is a lattice. Then we have a G -invariant measure on G/Γ so we may consider the L^2 space on it. Then

$$L^2(G/\Gamma)^{\mathcal{U}} \cong \text{Hom}_G \left(\frac{G}{\mathcal{U}}, L^2(G/\Gamma) \right)$$

duality principle

$$\mathrm{Hom}_{\Gamma}\left(\frac{G}{\mathcal{U}},\mathbb{R}^2\right)\rightarrow \mathrm{Hom}_{\mathcal{U}}\left(\frac{G}{\Gamma},\mathbb{R}^2\right)$$

$$\Phi\mapsto \tilde{\Phi}(g):=g\Phi\left(g^{-1}\right)?$$

A Pirkovskii - Harmonic Analysis and Banach Algebras

Harmonic Analysis and Banach Algebras. A. Pirkovskii. Fall 2024.

* Abstract

Harmonic analysis on locally compact abelian groups is a natural generalization of the classical Fourier analysis usually studied by undergraduate students in mathematics (that is, of the theory of trigonometric Fourier series and of the Fourier transform on the real line). The most elegant approach to harmonic analysis on abelian groups is based on the theory of commutative Banach algebras, which was initiated by Gelfand in the early 1940s and was further developed by Raikov, Naimark, Shilov and many other brilliant mathematicians. This approach, in particular, yields a relatively simple analytic proof of the Pontryagin duality based on the Plancherel theorem. In this course, we discuss the basics of Banach algebra theory and apply it to constructing the harmonic analysis on a locally compact abelian group. If time permits, some nonabelian groups will also be considered.

the circle

The Fourier series of $f \in L^2(S^1)$ states

$$f = \sum_{k \in \mathbb{Z}} \hat{f}(k) \chi_k$$

where

$$\begin{aligned} \chi_k : S^1 &\rightarrow S^1 \\ z &\mapsto z^k \end{aligned}$$

$$\chi_k(ab) = \chi_k(a) \chi_k(b)$$

so

$$\chi_k \in \text{HomTopGrp}(S^1, S^1)$$

 **Definition.** If G is an Abelian topological group, we say the Abelian group of all homomorphisms $G \rightarrow S^1$

$$\hat{G} := \text{HomTopGrp}(G, S^1)$$

that is, the group of all unitary characters of G , is the **dual group** of G .

Thus we may write

$$f = \sum_{\chi \in \hat{G}} \hat{f}(\chi) \chi$$

as we shall show that all unitary characters of S^1 are of the form χ_k for some $k \in \mathbb{Z}$.

overview

A representation

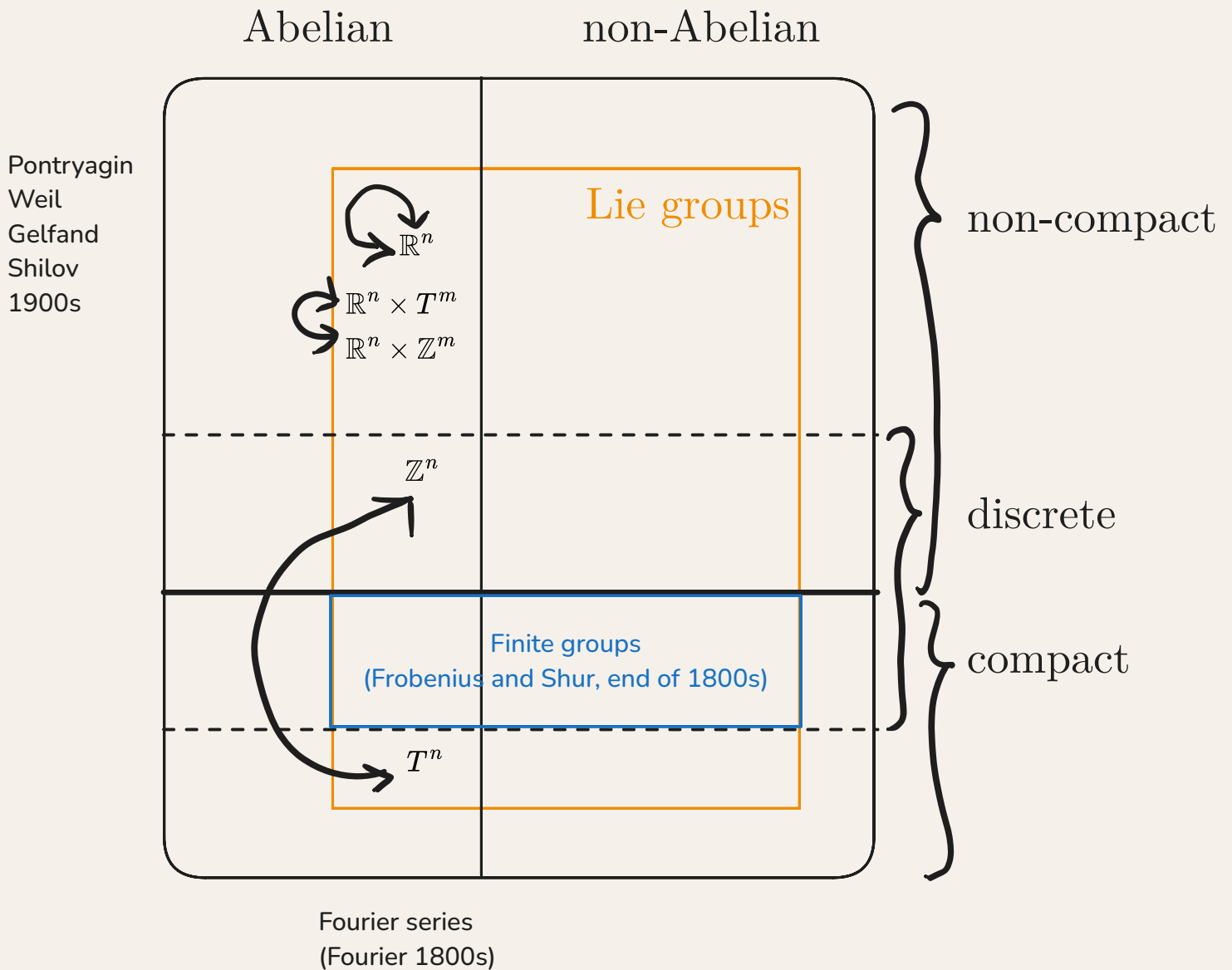
$$G \rightarrow GL(V)$$

makes

$$G \curvearrowright L^2(V)$$

 **Harmonic analysis and representation theory are closely related**

representation theory of locally compact groups



finite Abelian groups

For finite Abelian group G

$$\hat{G} = \text{HomGrp}(G, S^1)$$

are the set of unitary characters of

☰ If $G = \langle x \rangle$ is a finite cyclic group then

$$\begin{aligned} \hat{G} &\rightarrow \mu_n := \{z \in \mathbb{C} \mid z^n = 1\} \\ \chi &\mapsto \chi(x) \end{aligned}$$

is an isomorphism.

So $G \cong \hat{\hat{G}}$ is G is cyclic but not canonically.

Proposition: Let G, G_1, G_2 be finite Abelian groups

- $$\widehat{G_1 \times G_2} = \hat{G}_1 \times \hat{G}_2$$

- $x \in G, x \neq i_G$ implies

$$\exists \chi \in \hat{G} : \chi(x) \neq 1_{S^1}$$

- ("Pontryagin duality" for finite Abelian groups)

$$\begin{aligned} G &\cong \hat{\hat{G}} \\ x &\mapsto \text{ev}_x \end{aligned}$$

- $$\chi \in \hat{G}, \chi \neq 1 \implies \sum_{x \in G} \chi(x) = 0$$

Definition. Fourier transform for a finite Abelian group

The **Fourier transform** for a finite Abelian group is

$$\begin{aligned} \hat{}: \mathbb{C}^G &\rightarrow \mathbb{C}^{\hat{G}} \\ f &\mapsto \hat{f} \end{aligned}$$

where

$$\hat{f}(\chi) := \sum_{x \in G} f(x) \chi(x)$$

Proposition:

- $$\hat{\delta}_x = \text{ev}_x$$

Quote

Speaker: Utsav Dewan (ISI Kolkata, Stat Math Unit)

Time: Friday, 31/1/25, 5PM

Place: AB2-2A

Title: Boundary exceptional sets for radial limits of positive superharmonic functions on Harmonic manifolds

Abstract: By classical Fatou (resp. Littlewood) type theorems in various setups, it is well-known that positive harmonic (resp. superharmonic) functions have non-tangential (resp. radial) limits at almost every point on the boundary. In this talk, in the setting of non-positively curved Harmonic manifolds of purely exponential volume growth, we will see some size estimates of the exceptional sets of points on the boundary at infinity, where a suitable function blows up faster than a prescribed growth rate, along radial geodesic rays, in terms of its Hausdorff dimension or Hausdorff outer measure.

https://en.wikipedia.org/wiki/Fatou%27s_theorem

[2309.05661] [Boundary exceptional sets for radial limits of superharmonic functions on non-positively curved Harmonic manifolds of purely exponential volume growth](#)

[Colloquium: The Fourier transform on harmonic manifolds by Kingshook Biswas.](#)

Harmonic analysis on/geometry of homogeneous spaces



#reference

- [Analysis on Homogeneous Spaces Class Notes Spring 1994](#)
- [88-871 \(Dynamics of Group Actions\)](#)
- [Dynamics on homogeneous spaces](#)
- [M Bachir Bekka, Matthias Mayer - Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous spaces \(2000\).pdf](#)
- [Geom-Homog.pdf](#)

§1. Harmonic Analysis on Homogeneous Spaces

1. General Problems

Let X be a locally compact Hausdorff space acted on transitively by a locally compact group G . We assume G leaves invariant a positive measure μ on X . Let T_x denote the (unitary) representation of G on the Hilbert space $L^2(X)$ defined by $(T_x(g)f)(x) = f(g^{-1} \cdot x)$ for $g \in G$,

Chapter 1: Integral geometry and Radon transforms

Invariant measures on coset spaces

Definition. Invariant forms

Let M be a smooth manifold and

$$\Phi : M \rightarrow M$$

be a diffeomorphism. A differential form $\omega \in \Omega(M)$ is **Φ -invariant** if

$$\Phi^* \omega = \omega$$

Definition. Left and right invariant forms on a Lie group

Let G be a Lie group with Lie algebra $\mathfrak{g}$. A differential form $\omega \in \Omega(G)$ is **left-invariant** if

$$x \in G \implies L_x^* \omega = \omega$$

and **right-invariant** if

$$x \in G \implies R_x^* \omega = \omega$$

Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$ (left-invariant vector fields). Then we have n many 1-forms

$$\omega_1, \dots, \omega_n$$

such that

$$\omega_i(X_j) = \delta_{ij}$$

These 1-forms are left-invariant and thus

$$\lambda := \omega_1 \wedge \dots \wedge \omega_n$$

is also a left-invariant n -form on G and it is the unique left-invariant n -form on G up to a constant factor.

Radon transform on $\mathbb{R}^n$

Let

$$P(\mathbb{R}^{n*}) \cong \mathbb{R}P^{n-1}$$

be the compact smooth $n - 1$ -manifold of all hyperplanes in $\mathbb{R}^n$ passing through origin.

Each hyperplane ξ can be written

$$\xi = \{x \in \mathbb{R}^n \mid \langle x, \hat{n} \rangle = \alpha\}$$

This produces a *surjective* map onto the space of all hyperplanes $\mathbf{Pl}^n$ in $\mathbb{R}^n$

$$\begin{aligned} S^{n-1} \times \mathbb{R} &\rightarrow \mathbf{Pl}^n \\ (\hat{n}, \alpha) &\mapsto \xi = \{x \in \mathbb{R}^n \mid \langle x, \hat{n} \rangle = \alpha\} \end{aligned}$$

that is a smooth double covering(?) as

$$(\hat{n}, \alpha), (-\hat{n}, -\alpha)$$

produce the same plane. Thus we identify

$$\mathcal{C}^\bullet(\mathbf{Pl}^n) \leftrightarrow \{f \in \mathcal{C}^\bullet(S^{n-1} \times \mathbb{R}) \mid f(\hat{n}, \alpha) = f(-\hat{n}, -\alpha)\}$$

🔑 Definition. Radon transform and its dual

Let

$$f : \mathbb{R}^n \rightarrow \mathbb{C}$$

be integrable on each hyperplane in $\mathbb{R}^n$. Let $\mathbf{P}^n$ denote the compact smooth n -manifold of all hyperplanes in $\mathbb{R}^n$.

The **Radon transform** of f is

$$\begin{aligned} J(f) : \mathbf{P}^n &\rightarrow \mathbb{C} \\ \xi &\mapsto \hat{J}(f)(\xi) := \int_{(\xi, m_{\mathbb{R}^2})} f \end{aligned}$$

and the **dual Radon transform** of $\phi \in \mathcal{C}(\mathbf{P}^n)$

$$\begin{aligned} \mathbb{R}^n &\rightarrow \mathbb{C} \\ x &\mapsto \check{J}(\phi)(x) := \int_{x \in \xi} \phi \end{aligned}$$

Is every smooth manifold homogeneous?

- Given any two points in a connected manifold, there exists a diffeomorphism taking one to the other. ^[1]
- However, not all smooth manifolds are homogeneous G -spaces for (finite dim) Lie groups G , there are necessary conditions ^[2]

 **(Mostow, 2005)** M is a compact homogeneous G -space $\implies \chi(M) \geq 0$ ^[3]

Is a quotient of a Lie-homogeneous again Lie-homogeneous?

No. By

 **(Mostow, 2005)** M is a compact homogeneous G -space $\implies \chi(M) \geq 0$ ^[1]

1. [A Structure Theorem for Homogeneous Spaces | Geometriae Dedicata](#) ↩

higher genus surfaces are not Lie-homogeneous, but their universal cover is $\mathbb{R}^2$ which is Lie-homogeneous.

The fact that they are not Riemannian homogeneous is easily proven by Riemann surface techniques. [4]

Is every Lie-homogeneous smooth manifold Riemannian homogeneous?

No. [5]

 Quote

For example, consider $G = PGL(2, \mathbb{C})$ acting on the Riemann sphere S^2 via *Möbius transformations*. Suppose there is a G -invariant distance on S^2 . Let $H \subseteq G$ denote the isotropy group at $p := (1, 0, 0)$. That is, for $h \in G$, $h \in H$ iff $h * p = p$. For any $\epsilon > 0$, because G preserves distance, H will preserve the set $\{q \in S^2 : d(p, q) = \epsilon\}$. For small ϵ , this set should be a circle. However, since G acts 3-transitively on S^2 , it follows that H acts transitively on $S^2 \setminus \{p\}$, so H cannot preserve any circle. [6]

 Quote

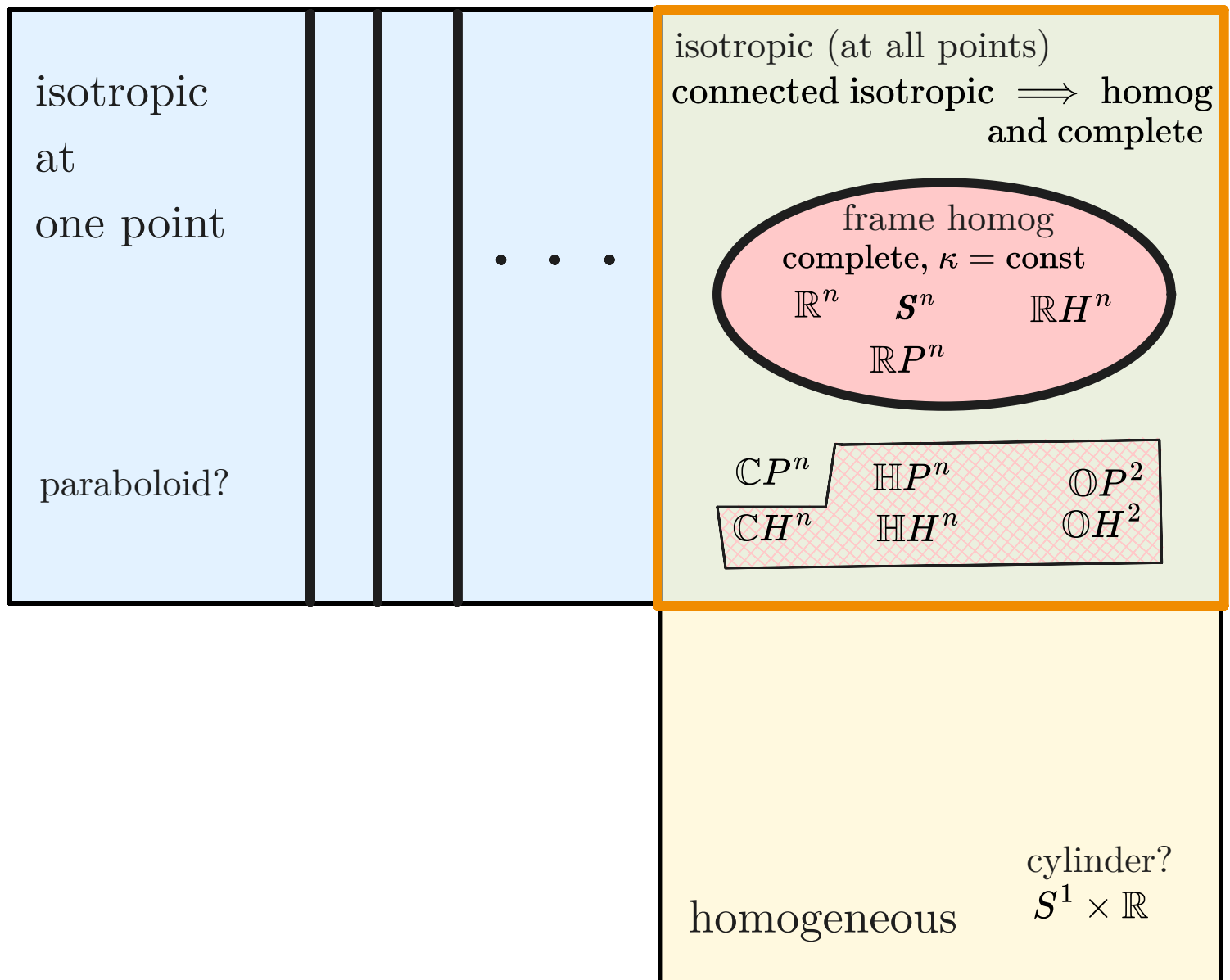
If a **compact** Lie group G acts transitively on a smooth manifold, then by averaging one can produce a Riemann metric on that manifold which is G -invariant. [7] [6-1]

Classification of connected isotropic Riemannian manifolds

[8]

[9]

connected Riemannian manifolds



8.12.2 Theorem. *Let M be a riemannian symmetric space. Then the following conditions are equivalent.*

- (i) *M is two point homogeneous.*
- (ii) *Either M is a euclidean space or M is irreducible and of rank 1.*
- (iii) *M is isometric to one of the spaces*
 - (a) $\mathbf{R}^n = \mathbf{E}(n)/\mathbf{O}(n)$, euclidean space,
 - (b) $S^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$, sphere,
 - (c) $\mathbf{P}^n(\mathbf{R}) = \mathbf{SO}(n+1)/\mathbf{O}(n)$, real projective space,
 - (d) $\mathbf{P}^n(\mathbf{C}) = \mathbf{SU}(n+1)/\mathbf{U}(n)$, complex projective space,

- (e) $\mathbf{P}^n(\mathbf{Q}) = \mathbf{Sp}(n+1)/\mathbf{Sp}(n) \times \mathbf{Sp}(1)$, *quaternionic projective space*,
 (f) $\mathbf{P}^2(\mathbf{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$, *Cayley projective plane*,
 (b') = (c') $\mathbf{H}^n(\mathbf{R}) = \mathbf{SO}^1(n+1)/\mathbf{SO}(n)$, *real hyperbolic space*,
 (d') $\mathbf{H}^n(\mathbf{C}) = \mathbf{SU}^1(n+1)/\mathbf{U}(n)$, *complex hyperbolic space*,
 (e') $\mathbf{H}^n(\mathbf{Q}) = \mathbf{Sp}^1(n+1)/\mathbf{Sp}(n) \times \mathbf{Sp}(1)$, *quaternionic hyperbolic space*,
 (f') $\mathbf{H}^2(\mathbf{Cay}) = \mathbf{F}_4^*/\mathbf{Spin}(9)$, *Cayley hyperbolic plane*.

Here ' denotes noncompact dual.

Classifying homogeneous Riemannian manifolds

Much harder.

	Homogeneous G -space M (that is $G \curvearrowright M$ transitively)	$M \cong_G \frac{G}{H}$ where H is a closed subgroup
compact G ?	G -invariant metrics on M	H -invariant inner products on $\mathfrak{g}/\mathfrak{h}$?

Isometric embedding of Riemannian 2-manifolds into $(\mathbb{R}^n, \text{DOT})$

[10]

1. [differential geometry - Proving that given any two points in a connected manifold, there exists a diffeomorphism taking one to the other - Mathematics Stack Exchange](#) ↩
2. [dg.differential geometry - Example of a manifold which is not a homogeneous space of any Lie group - MathOverflow](#) ↩
3. [A Structure Theorem for Homogeneous Spaces | Geometriae Dedicata](#) ↩
4. <https://mathoverflow.net/a/104106> ↩
5. <https://mathoverflow.net/questions/346364/is-every-homogeneous-space-riemannian-homogeneous> ↩
6. <https://math.stackexchange.com/a/3876991/1290493> ↩ ↩
7. [dg.differential geometry - How do you see that higher genus surfaces are not homogeneous? - MathOverflow](#) ↩
8. <https://math.stackexchange.com/a/4920625> ↩
9. <https://math.stackexchange.com/a/4921317> ↩
10. [dg.differential geometry - Construct embedding given metric - MathOverflow](#) ↩

Harmonic analysis on/geometry of homogeneous spaces



#reference

- [Analysis on Homogeneous Spaces Class Notes Spring 1994](#)
- [88-871 \(Dynamics of Group Actions\)](#)
- [Dynamics on homogeneous spaces](#)
- [M Bachir Bekka, Matthias Mayer - Ergodic Theory and Topological Dynamics of Group Actions on Homogeneous spaces \(2000\).pdf](#)
- [Geom-Homog.pdf](#)

TIFR-Geometry, groups and dynamics (2017)

- [Geometry, Groups and Dynamics \(GGD\) - 2017 - YouTube](#)
- [Geometry, Groups and Dynamics \(GGD\) - 2017 | ICTS](#)

Course Outline

The program focuses on geometry, dynamical systems and group actions. Topics are chosen to cover the modern aspects of these areas in which research has been very active in recent times. The program will have two parts, a preschool in the first week and an advanced school during the remaining two weeks. The pre-school will be devoted to covering the necessary background material for the advanced school during the remaining two weeks. The courses for the first week of pre-school include:

1. Discrete subgroups of Lie groups, by Pralay Chatterjee (IMSc, Chennai)
2. Introduction to hyperbolic geometry, by Subhojoy Gupta (IISc, Bangalore)
3. Introduction to Geometric Group Theory, by Pranab Sardar (IISER, Mohali)
4. Crash course on Riemannian geometry, by Harish Seshadri (IISc, Bangalore)
5. Teichmüller Theory, old and new, by Athanase Papadopoulos (Univ. Strasbourg, France)

More details of the courses will be given soon on this webpage.

The advanced school will consist of courses aimed at graduate students and young researchers who are either working or want to work in the broad theme of the program. The courses will be supplemented by a few research level surveys and lectures by some eminent experts in the field. These supplementary lectures are meant to connect the topics of the advanced school to cutting edge research to open new horizons of research for the participants of the school.

The following courses and mini courses for the two-week advanced school have been planned.

1. Riemann Surfaces and the Absolute Galois Group, by Norbert A'Campo (Univ. Basel, Switzerland)
2. Hierarchically hyperbolic space, by Jason Behrstock (CUNY, USA)
3. Projection Complexes, by Mladen Bestvina (Univ. Utah, USA)
4. Geodesic flow of the Weil-Petersson metric, by Keith Burns (Northwestern Univ., USA)
5. Growth in groups, by Rostislav Grigorchuk (TAMU, USA)
6. Higher Teichmüller Theory, by Francois Labourie (Univ. Nice, France) and Misha Kapovich (Univ. Davis, USA)
7. Boundary theory of hyperbolic groups, by Mahan Mj. (TIFR, India)
8. Geometrically infinite Kleinian groups, by Ken'ichi Ohshika (Osaka Univ., Japan)
9. Teichmüller Theory, old and new, by Athanase Papadopoulos (Univ. Strasbourg, France)
10. Deformation of complex hyperbolic lattices, by Pierre Will (Univ. Grenoble, France)

There will be some expository lectures as well. The program is followed by ICTS discussion meeting : [Surface Group Representations and Geometric Structures](#).

Talks

Week - 1: Basics on Geometry and Groups

1. **Norbert A'Campo**

Riemann surfaces: algebra, analysis, geometry

Riemann started the study of the possibility of finding meromorphic functions with prescribed truncated Laurent expansions at its zero's and poles on compact Riemann surfaces. This study was seminal and has forced discoveries in many, perhaps all, branches of mathematics. Here only some key words: topology, manifold, differential forms, cohomology, hyperbolic geometry, Gauss-Bonnet theorem, harmonic analysis, fundamental group, Teichmueller space,

Chow's Theorem, ... The lectures will have as goal the so-called Riemann Existence Theorem, Uniformisation Theorem for compact connected Riemann surfaces of genus $g > 1$, the Universal Curve,.....

2. Pralay Chatterjee

Symmetric spaces

Starting with the basics of symmetric spaces and examples, we will explore Chapter 2 of the book "Geometry of nonpositively curved manifolds" by P. Eberlein which deals with the structure of symmetric spaces of non-compact type.

3. Subhojoy Gupta

Hyperbolic surfaces and their Teichmüller spaces

In the first talk, we shall introduce the Teichmüller space T of a compact oriented surface S as the deformation space of hyperbolic structures on S . In the second talk, we shall discuss the relation with surface group representations to $PSL(2, \mathbb{R})$ and in the case S is closed, identify T with a component comprising discrete and faithful representations. In the final talk, we shall introduce the mapping class group of S and discuss its action on Teichmüller space T .

I will only assume some basic familiarity with hyperbolic plane and its isometries.

4. Mahan Mj

Introduction to hyperbolic groups

Hyperbolic metric spaces were discovered by Gromov in the 80's to give a unified treatment of manifolds of negative curvature and discrete groups satisfying certain combinatorial conditions. We shall describe various equivalent notions of hyperbolicity. We shall also describe the Gromov boundary of such spaces. (Reference: Bridson-Haefliger--Spaces of non-positive curvature).

5. Ken'ichi Ohshika

Boundaries of quasi-Fuchsian spaces and continuous/discontinuous phenomena

In this series talks, I shall first explain what kind of Kleinian groups appear on boundaries of quasi-Fuchsian spaces. Relying on examples given in the first part, I shall next show that as quasi-Fuchsian groups converge to a Kleinian group on the boundary, there are things which vary continuously and other things which vary discontinuously. In the last part I shall illustrate that this difference can be understood using geometric limits.

6. Athanase Papadopoulos

Lecture 1: Introduction to spherical geometry I

Lecture 2: Introduction to spherical geometry II

I will prove some basic theorems of spherical geometry : In a triangle, the angle sum is greater than two right angles ; the segment joining the midpoints of the two legs is greater than half of the basis, and I will give two formulas for the area (one in terms of angles, and one in terms of side lengths). I will discuss the relation with hyperbolic geometry, and more generally with Funk and Hilbert geometries.

7. Pranab Sardar

Geometry of the symmetric space $SL(n, \mathbb{R})/SO(n, \mathbb{R})$

I will discuss the geometry of $SL(n, \mathbb{R})/SO(n, \mathbb{R})$ following the book Metric Spaces of Non-positive Curvature by Bridson-Haefliger (Part II, Chapter 10). Starting from the definition of the metric we will prove that it is a CAT(0) space; we will describe the flats, and Weyl chambers etc. Time permitting we will touch on the Tits boundary of this space.

8. Harish Seshadri

Crash course in Riemannian geometry

I will discuss some basic techniques and results in Riemannian geometry relevant to the study of negatively curved manifolds. These will include Jacobi fields, the Cartan-Hadamard theorem and the Cartan-Ambrose-Hicks theorem.

The text "Riemannian Geometry" By Gallot, Hulin and Lafontaine will be the reference.

Week - 2: Concentration Period: Geometry and Groups

1. Jason Behrstock

Hierarchy Hyperbolic Spaces

In the first lecture we will introduce the mapping class group (MCG) of a surface and discuss the curve complex of a surface. This leads to several very powerful tools developed by Masur, Minsky, Behrstock, and others which we will survey. In the second lecture we will discuss fundamental groups of compact cube complexes, focussing in particular on the interesting special case of right angled Artin groups (RAAGs). After introducing these groups, we will begin to discuss recent tools developed by Behrstock-Hagen-Sisto which are analogues of the machinery discussed for the mapping class group in the first lecture; we will give details of the tools for RAAGs and describe the general framework of a hierarchically hyperbolic space (HHS). In the third lecture we will outline some of the techniques for working with HHS and give a sketch of the proof of some applications, including the quasi-flats theorems, which proves a number of outstanding conjectures.

References:

1. This is a survey which is a good introduction to the topics which will be covered in this mini-course: "What is a hierarchically hyperbolic space?", by Alessandro Sisto, <https://arxiv.org/abs/1707.00053>.
2. This paper won't be discussed explicitly, but it contains and applies a number of the topics discussed in the first lecture, so is a good reference to be aware of. "Geometry and rigidity of mapping class groups", by Jason Behrstock, Bruce Kleiner, Yair Minsky, and Lee Mosher, <https://arxiv.org/abs/0801.2006>
3. The second lecture will cover several of the topics of this paper in the special case of RAAGs: Hierarchically hyperbolic spaces I: curve complexes for cubical groups, by Jason Behrstock, Mark F. Hagen, and Alessandro Sisto, <https://arxiv.org/abs/1412.2171>
4. The main results of this paper will be discussed in lecture 3: "Quasiflats in hierarchically hyperbolic spaces", by Jason Behrstock, Mark Hagen, and Alessandro Sisto, <https://arxiv.org/abs/1704.04271>

2. Mladen Bestvina

Constructing group actions on quasi-trees and applications

Starting from relatively simple axioms one can construct a quasi-tree in a natural way. These axioms turn out to be satisfied in a variety of situations that arise in geometric group theory, most notably in the setting of Masur-Minsky subsurface projections, or in the presence of rank 1 group elements. The original construction is presented in arXiv:1006.1939, and it is joint work with Ken Bromberg and Koji Fujiwara. There is a simplification of the construction which I will present in the minicourse, and it is a joint work with Bromberg, Fujiwara, and Alessandro Sisto.

3. **Rostislav Grigorchuk**

Growth of finitely generated groups and related topics

I will explain what is the growth of a group (semigroup, algebra,...), the relation between growth of a group and of Riemannian manifold. Describe the main results about growth of groups from Shwartz and Milnor's contributions (50th-60th of 20th century) till our days. The main focus will be given to groups of intermediate growth (between polynomial and exponential). Starting from my original construction of such groups (that answered the question of Milnor and solved some other problems) I will finish by recent constructions by Bartholdi-Erschler and by Nekrashevych. Also such topics as actions on rooted trees, self-similar groups, automaton groups, amenable groups, and iterated monodromy groups will appear occasionally.

4. **Michael Kapovich**

Discrete subgroups of higher rank Lie groups

I will talk about recent advances in the theory of discrete subgroups of higher rank Lie groups, which exhibit some "rank one" behavior and related aspects of coarse geometry of higher rank symmetric spaces. This is based mostly on my work with Bernhard Leeb and Joan Porti.

5. **Francois Labourie**

Mini Course 1: Dynamics of Anosov representations

In this series of lectures I will give the definition of Anosov subgroups of linear groups emphasizing their dynamical nature. In particular, I will explain that they are associated to currents and that they have natural geodesic flows with dynamical properties. My talk intend to be elementary, starting from Anosov representations of the group Z .

Mini course 2: Introduction to Higgs bundles

In this series of lecture, I will first start with very elementary classical result on line bundles on Riemann surfaces. Following this historical point of view, I will move to the definition of Higgs bundles. My goal is to motivate and explain the definition and state the major theorems of the field.

6. **Ken'ichi Ohshika**

Boundaries of quasi-Fuchsian spaces and continuous/discontinuous phenomena

In this series talks, I shall first explain what kind of Kleinian groups appear on boundaries of quasi-Fuchsian spaces. Relying on examples given in the first part, I shall next show that as quasi-Fuchsian groups converge to a Kleinian group on the boundary, there are things which vary continuously and other things which vary discontinuously. In the last part I shall illustrate that this difference can be understood using geometric limits.

7. **Michah Sageev**

CAT(0) cube complexes and group theory

CAT(0) cube complexes are a particular class of CAT(0) spaces that carry a combinatorial structure which gives them the look and feel of "generalized trees". We will discuss the structure of CAT(0) cube complexes, what this structure tells us about the groups that act on them, and how to get build actions of groups on such complexes.

1. Anish Ghosh

Dynamical systems on homogenous spaces and number theory

I will survey some of S. G. Dani's many influential contributions to the above subject.

2. Athanase Papadopoulos

On some theorems on spherical geometry from Menelaus' Spherics (November 18 Special talk)

The « Spherics » by Menelaus of Alexandria (1st-2nd c. A.D.) is the most important book ever written on spherical geometry. It is a profound work. It contains 91 propositions, and some of them are very difficult to prove. An edition, from Arabic texts (the Greek original does not survive), is being published now by De Gruyter, in their series Scientia Graeco-Arabica, No. 21.

[Link](#)

This publication contains in particular the first English translation of Menelaus' treatise. In this talk, I will explain some of the major theorems on spherical geometry contained in this work.

Week-3: Concentration Period: Geometry and Dynamics

1. Jason Behrstock

Hierarchy Hyperbolic Spaces

In the first lecture we will introduce the mapping class group (MCG) of a surface and discuss the curve complex of a surface. This leads to several very powerful tools developed by Masur, Minsky, Behrstock, and others which we will survey. In the second lecture we will discuss fundamental groups of compact cube complexes, focussing in particular on the interesting special case of right angled Artin groups (RAAGs). After introducing these groups, we will begin to discuss recent tools developed by Behrstock-Hagen-Sisto which are analogues of the machinery discussed for the mapping class group in the first lecture; we will give details of the tools for RAAGs and describe the general framework of a hierarchically hyperbolic space (HHS). In the third lecture we will outline some of the techniques for working with HHS and give a sketch of the proof of some applications, including the quasi-flats theorems, which proves a number of outstanding conjectures.

References:

1. This is a survey which is a good introduction to the topics which will be covered in this mini-course: "What is a hierarchically hyperbolic space?", by Alessandro Sisto, <https://arxiv.org/abs/1707.00053>.
2. This paper won't be discussed explicitly, but it contains and applies a number of the topics discussed in the first lecture, so is a good reference to be aware of. "Geometry and rigidity of mapping class groups", by Jason Behrstock, Bruce Kleiner, Yair Minsky, and Lee Mosher, <https://arxiv.org/abs/0801.2006>
3. The second lecture will be cover several of the topics of this paper in the special case of RAAGs: Hierarchically hyperbolic spaces I: curve complexes for cubical groups, by Jason Behrstock, Mark F. Hagen, and Alessandro Sisto, <https://arxiv.org/abs/1412.2171>
4. The main results of this paper will be discussed in lecture 3: "Quasiflats in hierarchically hyperbolic spaces", by Jason Behrstock, Mark Hagen, and Alessandro Sisto, <https://arxiv.org/abs/1704.04271>

2. Keith Burns

Ergodicity of the Weil-Petersson geodesic flow

The Weil-Petersson metric is a Riemannian metric on the Teichmueller space of a Riemann surface. It is invariant under the action of the mapping class group and descends to a Riemannian metric with volume on the moduli space.

The Weil-Petersson metric has negative sectional curvatures but is incomplete. The curvature and its derivatives blow up as one approaches the boundary of Teichmüller space. The effect of negative curvature on the behaviour of the geodesic flow is well understood. In particular Hopf and Anosov showed that the geodesic flow of a compact manifold with negative curvatures is ergodic. Their results extend to the Weil-Petersson geodesic flow, but the incompleteness of the metric creates considerable additional difficulties.

The lectures will attempt to explain the arguments of Hopf and Anosov and to indicate the additional ideas needed to apply them to the Weil-Petersson geodesic flow.

3. **Shrikrishna Dani**

Hyperbolic geometry, the modular group and Diophantine approximation

Study of dynamics of the geodesic flow associated with the modular surface, consisting of the Poincaré plane viewed modulo the action of the modular group $SL(2, \mathbb{Z})$ acting as isometries, has been applied to study the distribution of values of quadratic forms at points on the Euclidean plane with integer coordinates. In these talks we shall discuss the framework and some results in this respect. Relation with some other questions in the area of Diophantine approximation will also be discussed.

4. **Anish Ghosh**

Dynamical systems on homogenous spaces and number theory

I will survey some of S. G. Dani's many influential contributions to the above subject.

5. **Francois Labourie**

Mini course 2: Introduction to Higgs bundles

In this series of lecture, I will first start with very elementary classical result on line bundles on Riemann surfaces. Following this historical point of view, I will move to the definition of Higgs bundles. My goal is to motivate and explain the definition and state the major theorems of the field.

6. **Athanase Papadopoulos**

Lecture 3 and 4 : The arc metric on Teichmüller space

I will give an overview of the arc metric on the Teichmüller space of surfaces with boundary, including a study of its geodesics, its horofunction boundary, and the fact that the arc metric converges in an appropriate way to the Thurston metric of surfaces without boundary.

Lecture 5 : Transitional geometry

I will give an introduction to transitional geometry, that is, the subject of continuous passages between geometries.

7. **Pierre Will**

Discrete groups in complex hyperbolic geometry

I will discuss the geometry of complex hyperbolic space, and describe examples of discrete subgroups of its isometry group. I will mainly focus on the complex hyperbolic analogues of quasi Fuchsian groups. My goal is to end up describing recent progresses concerning spherical CR structures on 3-manifolds, which appear on the boundary at infinity of quotients of the complex hyperbolic plane by discrete subgroups.

Harmonic Maps between surfaces and Teichmuller theory, I and II

We describe some of the basics of classical Teichmuller theory from the point of view of harmonic maps. The goal is to give an introduction to the Hitchin component of the $\mathrm{PSL}(2, \mathbb{R})$ character variety that one would undertake after one has identified a convenient setting and phrasing for the Hitchin equations, which in this case take the form of the 'Bochner equation' for harmonic maps of surfaces. We describe elements of the local deformation theory as well as the asymptotics. We will not assume any prior knowledge of harmonic maps.

Mahan Mj - Introduction to Hyperbolic groups

Example of hyperbolic groups (differential geometry): fundamental groups of closed negatively curved manifolds

For example let

$$\Gamma \curvearrowright H^n$$

act by isometries, properly discontinuously and cocompactly then

$$\frac{H^n}{\Gamma}$$

is a closed manifold.

Then

$$\pi_1 \left(\frac{H^n}{\Gamma} \right)$$

is a *hyperbolic group*.

Example of hyperbolic groups (combinatorial): combinatorial group theory

Small cancellation groups.

The above two very *different* examples were unified by Gromov (1982-87) to give the definition of *hyperbolic groups*.

Here, we study

Definition. Geodesic metric spaces

Metric spaces

$$(X, d)$$

with a **geodesic** between every pair of points, that is $\forall p, q \in X$ there exists an isometric embedding

$$\begin{aligned} [0, d(p, q)] &\rightarrow X \\ 0 &\mapsto p \\ d(p, q) &\mapsto q \end{aligned}$$

are called **geodesic metric spaces**.

Example of geodesic metric spaces (differential geometric): complete Riemannian manifolds

Example of geodesic metric spaces (discrete): connected graphs with d defined by edge length 1

Any such graph, by connectedness has a *geodesic* between every pair of points.

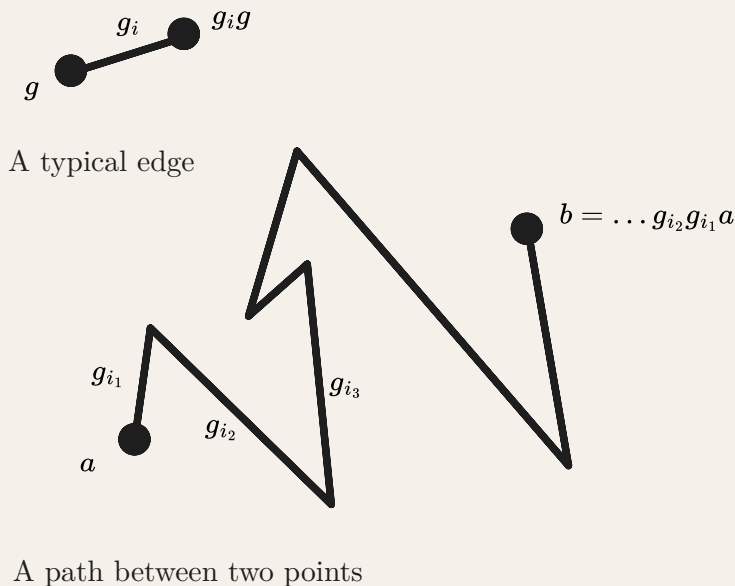
For example, the **Cayley graph of a finitely generated group**

$$G = \langle \underbrace{g_1, \dots, g_n}_{=:S} \mid r_i, i \in J \rangle$$

Then the Cayley graph

$$\Gamma(G, S) := \begin{cases} V := \{g \mid g \in G\} = G \\ E := \{(g, h) \mid g^{-1}h \in S\} \end{cases}$$

has edges between two points if they are related by a left multiplication by generators in S



Thus a path between $a, b \in G$ gives a word g_α in S that expresses b as

$$b = g_\alpha a$$

So length of this path is the number of edges. Thus distance $d(a, b)$ is the minimum length of word in S required to write $b = g_\alpha a$.

length spaces

Definition. Length of a path in a metric space

Let (X, d) be a metric space and

$$\gamma : I \subseteq \mathbb{R} \rightarrow X$$

be a curve where I is a non-empty interval ($\implies$ connected) set. Then **length of γ** is

$$\mathcal{L}(\gamma) := \sup \left\{ \sum_{i \in I} d(\gamma(t_{i-1}), \gamma(t_i)) \mid \text{all sequences } t_0 \leq t_1 \leq \dots \leq t_k \text{ in } I \right\}$$

We say γ is **rectifiable** if $\mathcal{L}(\gamma) < \infty$.

📌 Definition. Length metric of a metric space

Let (X, d) be a metric space. The **length metric** associated to d is the function

$$d_L : X \times X \rightarrow [0, \infty]$$
$$(x, y) \mapsto \inf\{\mathcal{L}(\gamma) \mid \gamma : [0, 1] \rightarrow X \text{ rectifiable path } \gamma(0) = x, \gamma(1) = y\}$$

The metric space (X, d) is a **length space** if $d = d_L$.

Proposition:

- The function d_L is a metric on $X \iff$ it is finite $\iff$ every pair of points in X can be joined by a rectifiable curve
 - In particular, a length space has this property.
- $d_L \geq d$
- $(d_L)_L = d_L$

📖 Example

For a connected Riemannian manifold (X, g) the induced metric space (X, d) is a **length space**.

Proposition: Every geodesic space is a length space.



As

$$d \leq d_L$$

and

$$d_L(x, y) \leq \mathcal{L}(\gamma_{x,y}) = d(x, y)$$

We conclude

$$d = d_L$$

📖 (Hopf-Rinow, Cohn Vossen 1935) Let X be a complete, locally compact length space. Then X is proper (every closed and bounded subset of X is compact) and a geodesic space.

quasi-isometries

🌀 Intuition

We need a notion of morphism that is **robust** enough so that $\Gamma(G, S)$ is independent of the choice of the generating set S

$$\Gamma(G, S_1) \xrightarrow{\text{Id}_G} \Gamma(G, S_2)$$

should be a morphism in the category of complete metric spaces we're working with. This is precisely **quasi-isometries**.

📌 Definition. Quasi-isometric embedding

A **K, ϵ -quasi-isometric embedding** between complete metric spaces is a map

$$f : (X, d) \rightarrow (Y, \rho)$$

if $\forall x_1, x_2 \in X$ we have

$$\frac{1}{K}d(x_1, x_2) - \epsilon \leq \rho(f(x_1), f(x_2)) \leq Kd(x_1, x_2) + \epsilon$$

Definition. Quasi-isometries

A **(K, ϵ, C) -quasi-isometry** of complete metric spaces

$$f : (X, d) \rightarrow (Y, \rho)$$

is a **(K, ϵ) -quasi-isometric embedding** and a C -coarsely surjective map, that is, the ball of radius C around the image is entirety of Y

$$Y = B_C(f(X))$$

Warning

Quasi-isometries *may* be discontinuous!

Slogan

- Calculus, differential geometry $\leftrightarrow$ good eyesight
- quasi-differential geometry $\leftrightarrow$ bad eyesight, coarse graining

Proposition: A (K, ϵ, C) -quasi-isometry f is equivalent to an (K, ϵ) -quasi-isometric embedding f and a (K', ϵ') quasi-isometric embedding such that

$$f \circ g, g \circ f$$

are C' -close to Id for some $C' > 0$, that is

$$d(g \circ f(x), x), d(f \circ g(y), y) \leq C'$$

Proposition: Let S_1, S_2 are finite symmetric ($g \in S_1 \iff g^{-1} \in S_1$) generating set of a group G . Then

$$\text{Id}_G : \Gamma(G, S_1) \rightarrow \Gamma(G, S_2)$$

(on the vertex set) is a **quasi-isometry**.

Take

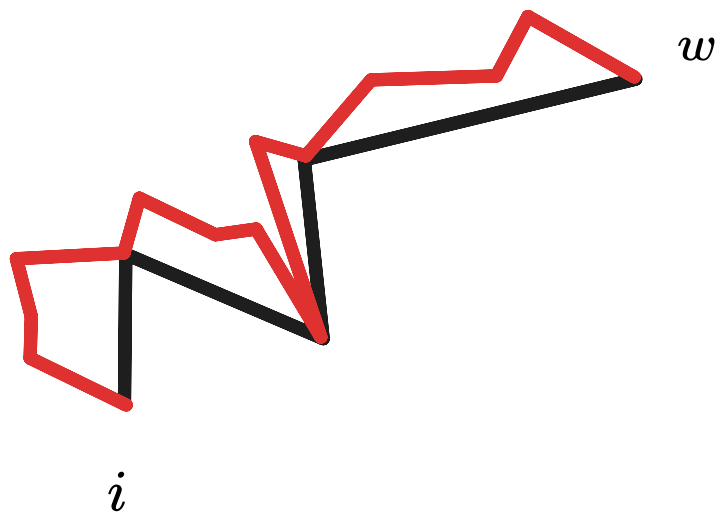
$$\begin{aligned} S_1 &= \{g_1, \dots, g_n\} \\ S_2 &= \{h_1, \dots, h_m\} \end{aligned}$$

- Then each h_i is a word in $\{g_1, \dots, g_n\}$ of some length l_i . Now

$$L := \max\{l_i\}$$

- This means

$$d_1(i, w) \leq Ld_2(i, w)$$



- Doing the opposite we have

$$d_2(i, w) \leq L' d_1(i, w)$$

- Hence we obtain a bi-Lipschitz map Id_G .

📌 **Definition.** A **geometric** group action is one that acts free, properly discontinuous, cocompact and by isometries.

📖 **(Milnor-Svarc lemma for complete Riemannian manifolds)** Let M be a complete Riemannian manifold and Γ be a finitely generated group acting **geometrically** on M . Then

$$(M, g) \cong_{\text{quasi}} \Gamma$$

Kathryn Mann - Dynamics of Group Actions

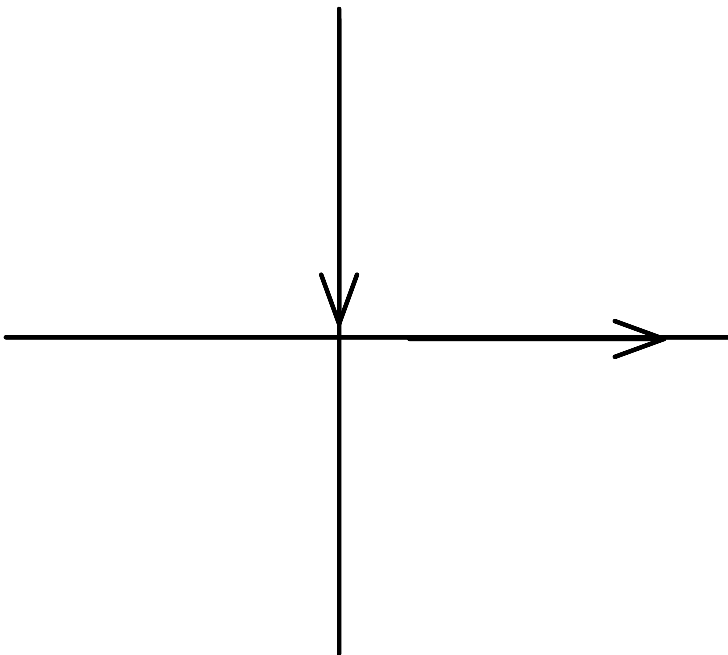
The broad theme of this minicourse is the relationship between the dynamics of a group acting on a space, the algebraic structure of the group, and some geometric structure responsible for the action; and how one of these can often determine the other two. A paradigm example of this is the "convergence group theorem" of Tukia, Gabai, and Casson--Jungreis, which says that a convergence group (a purely dynamical condition) acting on the circle is a Fuchsian group, i.e isomorphic to the fundamental group of a hyperbolic surface or orbifold; and the action on the circle is the action on the boundary of the surface or orbifold's universal cover.

In a similar spirit, Thurston defined an "extended convergence group" for groups acting on the line, and showed using ideas of Barbot and Fenley that these are always fundamental groups of 3-manifolds, and the action on the line comes from an Anosov flow on the 3-manifold. However, not all 3-manifolds with Anosov flows give rise to extended convergence groups; instead giving rise to a more general class called "Anosov-like actions on bi-foliated planes"

In the course, we'll study Anosov-like actions from the axioms, and see how some simple dynamical assumptions can lead to a rich structure theory. This is a perspective that Barthelmé, Bonatti, Fenley, Frankel and I (in various combinations and in still ongoing work) have recently successfully used towards the study and classification of Anosov flows on 3-manifolds, building on work of Barbot, Fenley, and others.

- geodesic flow on round n -sphere
 - trajectories periodic
 - global instability
- geodesic flows of n -manifolds with negative curvature
 - trajectories are "chaotic"
 - global stability (Anosov '60)
- "looks" like iterations of

$$\begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix}, \lambda > 1$$



Anosov flows

🔑 **Definition.** A flow on M is **Anosov** if $\exists$ continuous splitting

$$TM = \underbrace{X}_{\text{flow field}} \oplus \underbrace{E^{ss}}_{\text{contracted}} \oplus \underbrace{E^{uu}}_{\text{expanded}}$$

where contracted means

$$\exists C, K > 0 : v \in E^{ss}, t > 0 \implies \|\phi_*^t(v)\| < Ce^{-|t|K} \|v\|$$



🔑 **Definition.** A flow on M is **Anosov** if $\exists$ continuous splitting

$$TM = \underbrace{X}_{\text{flow field}} \oplus \underbrace{E^{ss}}_{\text{contracted}} \oplus \underbrace{E^{uu}}_{\text{expanded}}$$

where contracted means

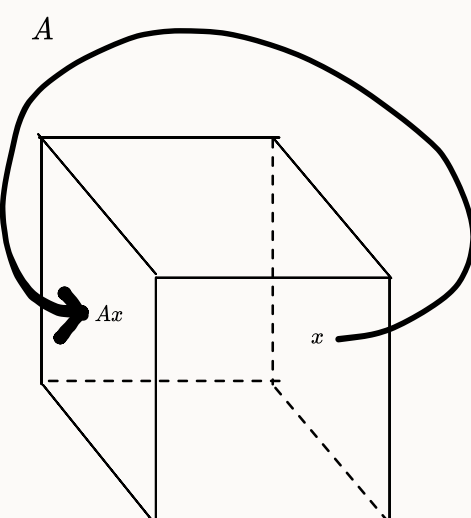
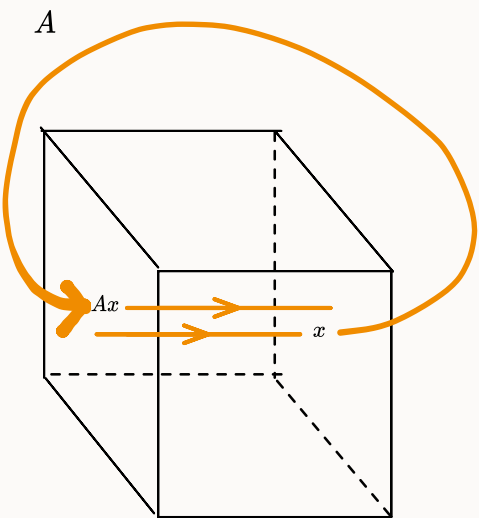
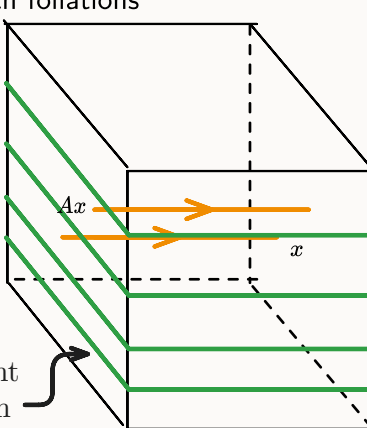
$$\exists C, K > 0 : v \in E^{ss}, t > 0 \implies \|\phi_*^t(v)\| < Ce^{-|t|K} \|v\|$$

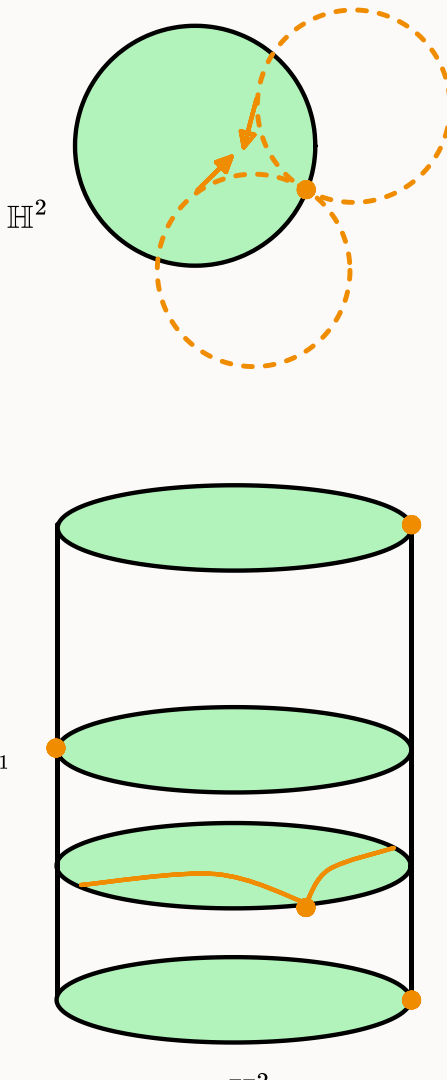
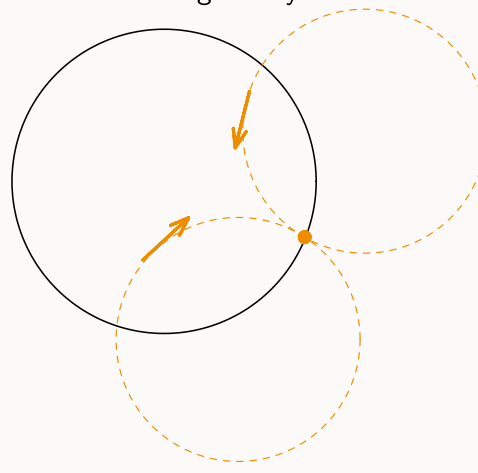
$X \oplus E^{ss}$ is tangent to some foliation $\mathcal{F}^s$ invariant under ϕ^t and similarly for $X \oplus E^{uu}$ is tangent to $\mathcal{F}^u$

🔑 **Definition.** Two flows ϕ^t, ψ^t on manifolds M_1, M_2 are **topologically orbit equivalent** if there is a homeomorphism $h : M_1 \rightarrow M_2$ that takes orbits of ϕ to orbit of ψ .

📖 Two Anosov flows that are **$\mathcal{C}^1$ -stable**, that is, if the two vector fields that generate the Anosov flows are $\mathcal{C}^1$ -close, then the flows are **topologically orbit equivalent**.

examples: suspensions and geodesic flows

manifold	flow	stable and unstable
$M := \frac{\mathbb{R}^2}{\mathbb{Z}^2} \times [0, 1]$ $(x, 1) \sim (Ax, 0)$  with $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ where the flow is	A  We may write the flow on $M = \frac{\mathbb{R}^2}{\mathbb{Z}^2} \times \mathbb{R}$ $(x, s) \sim (Ax, s - 1)$ as $\phi^t(x, s) := (x, s + t)$ 3M has a SOL (metric) structure and for this metric Anosov conditions hold with $C = 1$	here E^{ss} and E^{uu} are eigendirections of A with foliations  invariant foliation
unit tangent bundle $T^u(N)$ of a hyperbolic surface N		horocycle foliations

manifold	flow	stable and unstable
$T^u\mathbb{H}^2 = \mathbb{H}^2 \times S^1$ 		the foliation is given by 

$$\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{i}{2} & -\frac{i}{2} \end{bmatrix}$$

the classification problem

Proposition: stability $\implies$ upto orbit equivalence there are only countably many Anosov flows on a given compact smooth manifold

Classification problem of Anosov flows (Smale, '66)

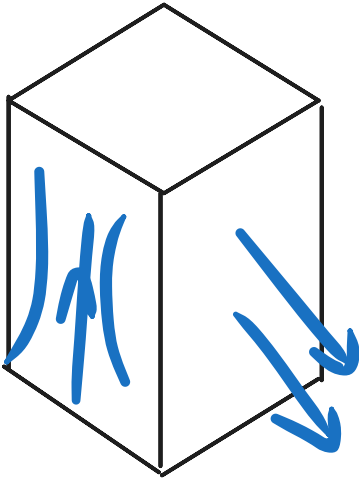
We hope to classify all Anosov flows

Proposition: stability $\implies$ upto orbit equivalence there are only countably many Anosov flows on a given compact smooth manifold

on all (countably many) compact smooth manifolds by discrete algebraic invariants.

- For $\dim > 3$, still very few examples known.
- For $\dim = 3$, there are many examples beyond suspensions and geodesic flows.
 - '70s-80s: New examples on 3-manifolds via "*surgery*"

- 80s Goodman: $(1, n)$ Dehn surgery procedure: cut and glue along annulus transverse to the flow
- Fried: Dehn surgeries along periodic orbits
- Bonatti Langevin: "building-block construction" define a flow on $[0, 1]^3$ and

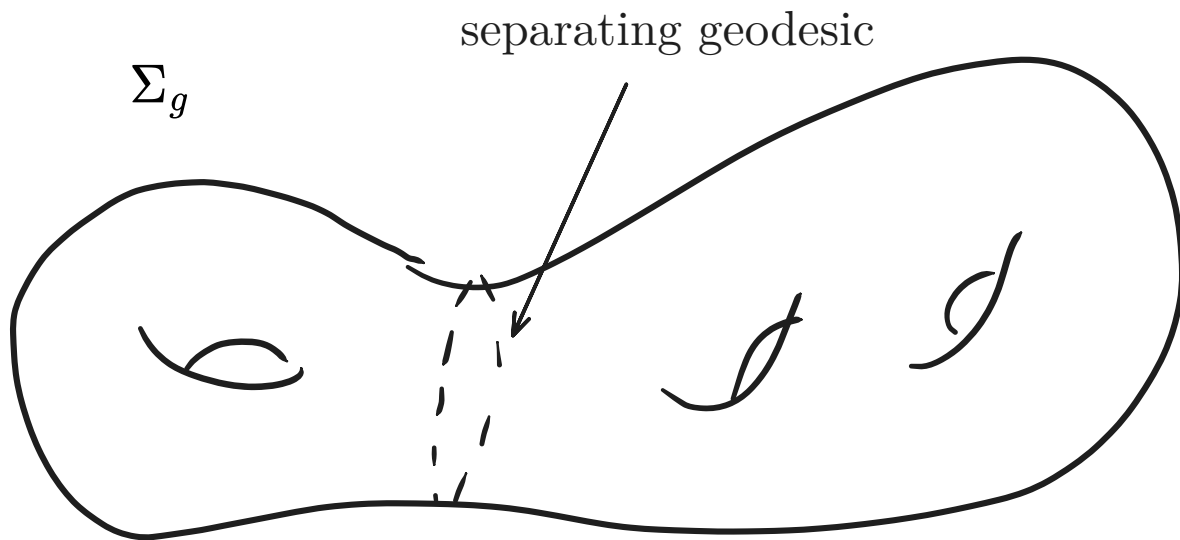


- More recently, Benjamin Bonatti-Yi: Graph manifolds: $\forall n$ there exists a compact Graph manifold with $\geq n$ distinct (orbit eqv) classes of Anosov flows
- Clay-Pinsky: different approach
- Boaden-Mann: $\forall n$ there exists a hyperbolic 3M with $\geq n$ Anosov flows

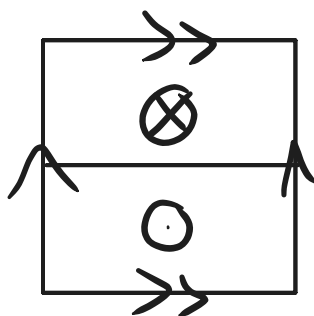
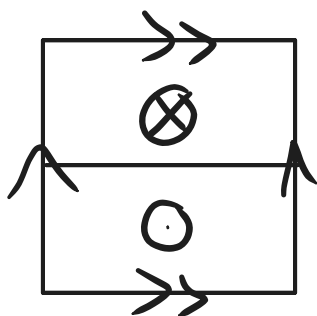
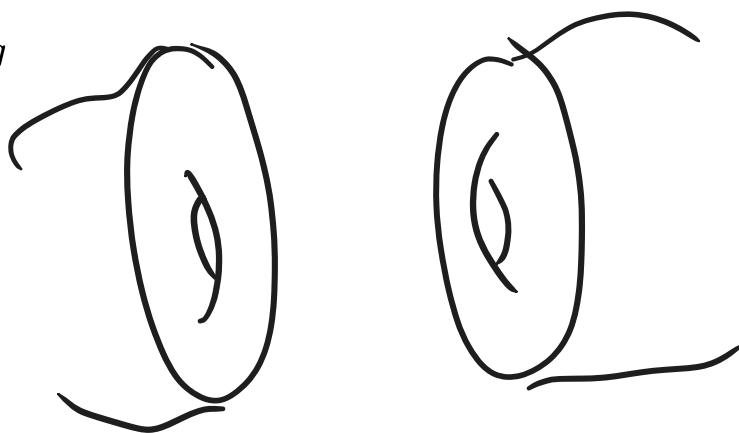
Classification problem

- Which 3 manifolds admit a Anosov flow?
 - obstructions are known: necessary conditions for 3M to have Anosov flows:
 - aspherical (have universal cover is contractible),
 - fundamental group has exponential growth
 - $\pi_1(^3M)$ is left orderable
 - ...
- How to classify all examples on a given manifold?
 -

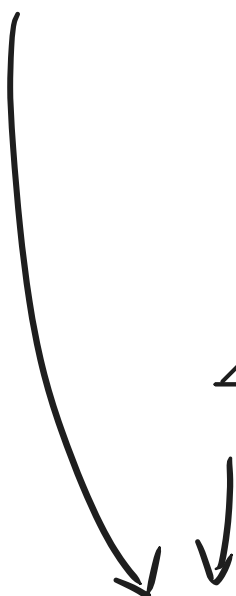
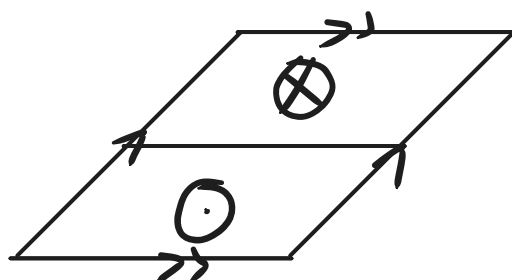
'80: *Anosov Flows on New Three Manifolds*, [Michael Handel](#) and [William P. Thurston](#)



$T^u \Sigma_g$



shear/Deck transformation



glue!

Here the manifold obtained is new, and the new flow is Anosov.

orbit space

 **Definition. Orbit space of a Anosov flow**

Given flow ϕ^t on 3M lift to $\tilde{\phi}^t$ on $\tilde{M}$ universal cover. Then **orbit space of ϕ^t** is

$$\mathcal{O}_\phi := \frac{\tilde{M}}{x \sim \phi^t(x)}$$

 **(Barbot, Fenley)** If ϕ^t is Anosov on M then the universal cover $\tilde{M} \cong \mathbb{R}^3$ and

$$\mathcal{O}_\phi \cong \mathbb{R}^2$$

Moreover $\pi_1({}^3M) \curvearrowright \tilde{M}$ (preserves $\tilde{\mathcal{F}}^s, \tilde{\mathcal{F}}^u$) descends to an action on $\mathcal{O}_\phi$ by homeomorphisms (which preserves induced foliations).

 **(Barbot)** Let ϕ^t, ψ^t are Anosov flows on 3M . They are orbit equivalent via a map $\Phi \iff \exists$ a homeomorphism

$$h : \mathcal{O}_\phi \rightarrow \mathcal{O}_\psi$$

that takes stable/unstable foliations to foliations such that

$$\forall \gamma \in \pi_1(M), h \circ \gamma = \Phi(\gamma) \circ h$$

references

<https://drive.google.com/file/d/1JMfXlAs6i6f8YXAERFWHz-HVhd0mrvPv/view>

1. [EUDML](#) | [Anosov Flows on New Three Manifolds.](#) 

Fields Institute - Geometry, Arithmetic, and Dynamics of Discrete Groups

Geometry, Arithmetic, and Dynamics of Discrete Groups | Fields Academy Shared Graduate Course - YouTube

Lie group action by a discrete subgroup	Riemannian geometry	dynamics	arithmetic
$\Gamma \overset{\text{discrete}}{\curvearrowright} X \cong G/K$ where $G = \text{Isom}(X)$ is a Lie group, $K = G_x$ is a cpt subgroup, $\Gamma \subset G \subset GL_{\mathbb{R}}(n)$	$(X, \tilde{g})$ then the quotient $\frac{X}{\Gamma} \cong_{\text{RiemOrbi}} (M, g)$ is a manifold if Γ is torsion free (then $\pi_1(M) = \Gamma$); tessellations of X by isometric copies of convex polyhedra	Behavior of Γ -orbits	Matrix entries /traces traces of Γ
$\mathbb{H}^n = PO_{\mathbb{R}}(n, 1)/O_{\mathbb{R}}(n)$		$\Gamma \curvearrowright \partial \mathbb{H}^n \cong S^{n-1}$ ergodicity?	
$\mathbb{R}^n = E_{\mathbb{R}}(n)/O_{\mathbb{R}}(n)$			
$\mathbb{Z} \curvearrowright \mathbb{R}$ $G = \mathbb{R} \ltimes \mathbb{Z}_2, K = \mathbb{Z}_2$	$M = S^1$		

ICTS '10 - Geometry Topology and Dynamics in Negative Curvature

<https://www.icts.res.in/sites/default/files/1280718981TitleAbstract.pdf>

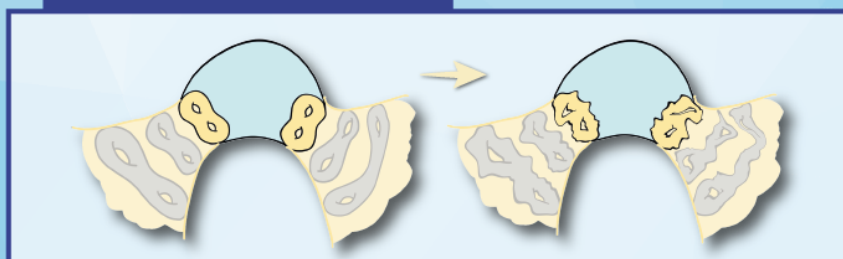
Martin Bridson

space	$K(\mathbb{Z}^n, 1)$	$K(\Sigma_g, 1)$	$K(F_n, 1)$
group	$SL(n, \mathbb{Z})$	$\mathrm{MCG}_g \cong \mathrm{Out}(\Sigma_g)$	$\mathrm{Out}(F_n)$
acts by	diffeomorphisms	diffeomorphisms	homotopy equivalences
classifying space?	$\mathbb{R}^? \cong \frac{SL(n, \mathbb{R})}{SO(n, \mathbb{R})}$	Teichmuller space $\mathcal{T}_g$ $\cong \mathbb{R}^{6g-g}$	outer space, a finite dimensional contractible space
which is a	complete CAT(0) manifold	incomplete CAT(0)	no non-positive curved metric
the group is generated by		Dehn twists	homotopy equivalences
		pseudo-Anosov	
group automorphisms			
quotients	$PSL(n, \mathbb{Z})$ or finite quotients like $SL(n, \mathbb{Z}_m)$	$1 \rightarrow \mathrm{Tor}_g \rightarrow \mathrm{MCG}_g \rightarrow Sp(2g, \mathbb{Z})$	Take characteristic $H < F_n$ we get $\mathrm{Out}(F_n) \rightarrow \mathrm{Out}\left(\frac{F_n}{H}\right)$ for example subgroups in lower central series?

IHEs '23 Homogeneous Dynamics and Geometry in Higher-Rank Lie Groups

– WORKSHOP –

HOMOGENEOUS DYNAMICS AND GEOMETRY IN HIGHER-RANK LIE GROUPS



**June
19 > 23
2023**

IHES

Marilyn and James Simons Conference Center
35 route de Chartres
91440 Bures-sur-Yvette

Organizers

Martin Bridgeman, *Boston College*
Richard Canary, *Univ. of Michigan*
Fanny Kassel, *CNRS & IHES*
Hee Oh, *Yale Univ.*
Maria Beatrice Pozzetti, *Univ. Heidelberg*
Jean-François Quint, *CNRS & Univ. de Bordeaux*



Information:
WWW.IHES.FR

Minicourses

- > **Jean-François Quint**, *CNRS & Univ. de Bordeaux*
“Dynamics on Higher-rank Lie Groups”
- > **Tengren Zhang**, *National Univ. of Singapore*
“Anosov Representations”
- > **Andrés Sambarino**, *CNRS & IMJ-PRG*
& **Minju Lee**, *Univ. of Chicago*
“Recent Developments”

Speakers

- > **Pierre-Louis Blayac**, *Univ. of Michigan*
- > **Leon Carvajales**, *Univ. de la Republica*
- > **Nguyen-Thi Dang**, *Univ. Paris-Saclay*
- > **Sam Edwards**, *Durham Univ.*
- > **François Labourie**, *Univ. Côte d'Azur*
- > **Sara Maloni**, *Univ. of Virginia*
- > **Giuseppe Martone**, *Yale Univ.*
- > **Wenyu Pan**, *Univ. of Toronto*
- > **Rafael Potrie**, *Univ. de la Republica*
- > **Pratyush Sarkar**, *UC San Diego*
- > **Andrew Zimmer**, *Univ. of Wisconsin*



STRUCTURES
CLUSTER OF
EXCELLENCE

L'IHES, membre fondateur de



Homogeneous Dynamics and Geometry in Higher-Rank Lie Groups

The goal of the workshop is to explore links between homogeneous dynamics and recent developments on infinite-covolume discrete subgroups of Lie groups, including images of Anosov representations and generalizations. There will be three minicourses and a dozen research talks.

This workshop is organized by **Martin Bridgeman** (Boston College), **Richard Canary** (University of Michigan), **Fanny Kassel** (CNRS & IHES), **Hee Oh** (Yale University), **Maria Beatrice Pozzetti** (Universität Heidelberg), and **Jean-François Quint** (CNRS & Université de Bordeaux).

Minicourses:

- **Jean-François Quint** (CNRS & Université de Bordeaux) - “Dynamics on Higher-rank Lie Groups” [3h]

- **Tengren Zhang** (National University of Singapore) - “*Anosov Representations*” [3h]
- **Andrés Sambarino** (CNRS & IMJ-PRG) & **Minju Lee** (University of Chicago) - “*Recent Developments*” [2+2h]

Research Talks:

- **Pierre-Louis Blayac** (University of Michigan)
- **León Carvajales** (Universidad de la Republica, Montevideo)
- **Nguyen-Thi Dang** (Université Paris-Saclay)
- **Sam Edwards** (Durham University)
- **François Labourie** (Université Côte d'Azur)
- **Sara Maloni** (University of Virginia)
- **Giuseppe Martone** (Yale University)
- **Wenyu Pan** (University of Toronto)
- **Rafael Potrie** (Universidad de la Republica, Montevideo)
- **Pratyush Sarkar** (UC San Diego)
- **Andrew Zimmer** (University of Wisconsin, Madison)

✓ https://www.youtube.com/playlist?list=PLx5f8IeIFRgENNBbT_A7c15Cbk1oaqEsF

frame: PDF

style: height: 900px;

urlSuffix: www.youtube.com/playlist?list=PLx5f8IeIFRgENNBbT_A7c15Cbk1oaqEsF

✓ <https://www.ihes.fr/~kassel/GroupsStAndrews-Kassel.pdf#page=2>

	Noncompact simple Lie group G	Maximal compact subgroup K	$\text{rank}_{\mathbb{R}}(G)$
A	$\text{SL}(n, \mathbb{C})$	$\text{SU}(n)$	$n - 1$
B	$\text{SO}(2n + 1, \mathbb{C})$	$\text{SO}(2n + 1)$	n
C	$\text{Sp}(2n, \mathbb{C})$	$\text{Sp}(n)$	n
D	$\text{SO}(2n, \mathbb{C})$	$\text{SO}(2n)$	n
A I	$\text{SL}(n, \mathbb{R})$	$\text{SO}(n)$	$n - 1$
A II	$\text{SU}^*(2n)$	$\text{Sp}(n)$	$n - 1$
A III	$\text{SU}(p, q)$	$\text{S}(\text{U}(p) \times \text{U}(q))$	$\min(p, q)$
BD I	$\text{SO}(p, q)_0$	$\text{SO}(p) \times \text{SO}(q)$	$\min(p, q)$
D III	$\text{SO}^*(2n)$	$\text{U}(n)$	$\lfloor n/2 \rfloor$
CI	$\text{Sp}(2n, \mathbb{R})$	$\text{U}(n)$	n
C II	$\text{Sp}(p, q)$	$\text{Sp}(p) \times \text{Sp}(q)$	$\min(p, q)$

Table 4.1 *List of classical noncompact simple real linear Lie groups, up to local isomorphism. Here $n, p, q \geq 1$ are integers. For types A, A I, and A II we assume $n \geq 2$, for types D and D III we assume $n \geq 3$, and for type BD I we assume $(p, q) \notin \{(1, 1), (2, 2)\}$.*

Jean-François Quint - An introduction to *geometry and dynamics* on semi-simple Lie groups

reminiscent of groups

G ^{connected} ~~reminiscent~~ Lie group

$\Leftrightarrow \mathfrak{g} = \text{Lie}(G)$ ~~reminiscent~~

$I \subset \mathfrak{g}$ ideal $[\mathfrak{g}, I] \subset I$

$\mathfrak{g}$ ~~reminiscent~~ $\Leftrightarrow \left(\begin{array}{l} \forall I \subset \mathfrak{g} \text{ ideal} \\ \text{abelian} \Rightarrow \{0\} = I \end{array} \right)$

$\mathfrak{g} \subset \mathfrak{gl}(V)$ V irreducible for $\mathfrak{g}$.
if $\forall W < V$ $\mathfrak{g}W \subset W$
 $\Rightarrow W = \{0\}$ or V

V totally reducible if $V = \bigoplus V_i$
 V_i irreducible

Ex $\mathfrak{g} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \mathfrak{gl}_2(\mathbb{R})$ $\mathbb{R}^2$ not totally reducible.

Thm: $\mathfrak{g} \subset \mathfrak{gl}(V)$

V irreducible

$\Rightarrow \mathfrak{g} = \mathbb{C} \oplus \mathbb{A}$

$\mathbb{A}$ abelian ($\cong \mathbb{R}$ or $\mathbb{R}^2$)

$[\mathbb{C}, \mathbb{A}] = \{0\}$

$\mathbb{A}$ ~~reminiscent~~

$\mathfrak{g} \subset \mathfrak{gl}(V)$

~~reminiscent~~

$\Rightarrow V$ totally reducible

$\mathfrak{g}$ ~~reminiscent~~

$\mathfrak{g} = \bigoplus \mathfrak{g}_i$

$\mathfrak{g}_i$ ideals
simple
(no proper nonzero ideals)

$$SL_n(\mathbb{R})$$

$$n \geq 2$$

$$SL_n(\mathbb{C})$$

$$n \geq 2$$

$$Sp_{2n}(\mathbb{R})$$

$$n \geq 1$$

$$Sp_{2n}(\mathbb{C})$$

$$n \geq 1$$

$$SO_n(\mathbb{C})$$

$$n \geq 3$$

$$SO(p, q)$$

$$p+q \geq 3$$

$$SU(p, q)$$

$$p+q \geq 2$$

$$Sp(p, q)$$

$$p+q \geq 1$$

$$SU^*(2n)$$

$$n \geq 2$$

$$SO^*(2n)$$

$$n \geq 2$$

classical
simple
lie algebras

+ finitely
many
other
exceptional

Raghunathan - Discrete subgroups of Lie groups

[#book](#) [M S Raghunathan - Discrete Subgroups of Lie Groups - Springer \(1972\).pdf](#) [#course/mini](#) [Discrete subgroups of Lie groups - YouTube](#)

Quote

It is well-known [20] that Γ embeds as a co-compact discrete subgroup of a simply connected nilpotent Lie group G , its Malcev closure ^[1]

1. <https://arxiv.org/pdf/1204.1613#page=4.00>



Rodriguez Hertz - Rigidity

[Rigidity - FRH - YouTube](#)

[project_rigiditynotes.pdf](#) from [Alp Uzman's Webpage](#)

Definition 10: Let G be an LCH group. G is **amenable** if one of the following equivalent conditions hold:

Federico Rodríguez Hertz
Alp Uzman

24

[redacted]
azu4@psu.edu

Penn State

Math 597.002: Rigidity

Spring 2021

- (i) (**Hulanicki condition**) The left regular representation of G has almost invariant vectors.
- (ii) (**Dynamical condition**) Any continuous action of G on a compact metric space by homeomorphisms has an invariant probability measure.
- (iii) (**Dynamical condition, abstract version**) Any continuous action of G on a nonempty compact convex subset of a separable locally convex topological vector space by affine automorphisms has a fixed point.
- (iv) (**Følner condition**) $\forall \varepsilon \in \mathbb{R}_{>0}, \forall K \in \mathcal{K}(G), \exists B \in \mathcal{B}(G), 0 < \lambda(B) < \infty, \forall k \in K : \lambda((kB) \triangle B) < \varepsilon \lambda(B)$. Here λ is the left invariant Haar measure on G and $A \triangle B := (A \setminus B) \uplus (B \setminus A)$ is the **symmetric difference**.

Summer School on Rigidity of Discrete Groups, June 30 – July 4, 2025 at IISER Mohali

(Supported by IISER Mohali and The Indian Mathematics Consortium)

[IISERM-Rigidity.docx](#) - [Google Docs](#)

Organisers:

Krishnendu Gongopadhyay (IISER Mohali) Email: krishnendu@iisermohali.ac.in	Pranab Sardar (IISER Mohali) Email: psardar@iisermohali.ac.in
--	--

In the proposed meeting, we shall focus on some aspects of the broad topic Rigidity in Geometry and Topology. The geometric form of Mostow's celebrated rigidity theorem states that every isomorphism between fundamental groups of closed hyperbolic manifolds of dimension at least 3, is induced by an isometry. This result was followed by a similar theorem of Prasad for nonuniform lattices. However, in a spectacular breakthrough, G. A. Margulis obtained a far reaching generalization of these theorems for irreducible lattices in higher rank groups. This result is now known as the Margulis' Superrigidity Theorem. The ideas behind proofs of these theorems have become very important and fruitful in modern mathematics. In the proposed workshop, we aim to provide a gentle introduction to these results along with similar rigidity theorem in geometry and topology through mini courses and survey talks.

Speakers: Marc Bourdon (U. Lille), Pralay Chatterjee (IMSc), Soma Maity (IISER Mohali), Arghya Mondal (IISER Mohali), C. S. Rajan (Ashoka), Pranab Sardar (IISER Mohali), Riddhi Shah (JNU), T. N. Venkataramana (ICTS Bengaluru).

Expected Participants: Faculty members, PhD students and postdoctoral fellows working in related areas. The topics to be covered in this school require some prior knowledge of geometric group theory, smooth manifolds and Lie groups.

Due to limited funding, we shall be able to provide local hospitality to only a limited number of participants who will be selected based on the proximity of their research interests with the theme of the summer school. Participants are requested to arrange their own travel funding.

Name of the Speaker	No. of Lectures	Abstract
Marc Bourdon (MB), Université de Lille, France	4 lectures	Title: Lie groups and quasi-isometries Abstract: In geometric group theory, one studies groups as geometric objects, and tries to determine whether an algebraic property is a geometric one. In this setting, the natural maps between groups are the so-called quasi-isometries (QI). A class of groups is said to be QI-rigid if every finitely generated group that is QI to a group in the class, is virtually isomorphic to (another) group in the class. Geometric group theory started in the 80's when Gromov proved that the class of nilpotent groups is QI-rigid. In the 90's the subject focused on semisimple groups; it culminated when a combination of works allowed to establish that the class of lattices in a given semisimple group is QI-rigid. Since then, geometric group theory has developed several interactions with other fields of mathematics. It has been used to solve some old open problems (e.g. in 3-manifold topology). A major problem in geometric group theory is to classify connected Lie groups up to QI. One knows that every simply connected Lie group is QI to a completely solvable Lie group, i.e. to a closed subgroup of the upper triangular real matrix group. In 2018 Cornulier conjectured that two QI completely solvable groups must be isomorphic. This is currently open, even in the smaller class of nilpotent Lie groups. During the talks, I plan to introduce some QI-invariants for Lie groups, and illustrate them with examples. These invariants include the growth rate, the rank, and the L^p-cohomology. The examples of groups that will be discussed include the nilpotent groups, the Heintze groups, the abelian-by-abelian solvable Lie groups. Some known results about

Name of the Speaker	No. of Lectures	Abstract
		their QI classification will also be presented.
Pralay Chatterjee (PC), IMSc Chennai	3 lectures	Title: Introduction to Lattices Abstract: Semisimple Lie groups, Lattices in Lie groups, Some generalities on lattices in Lie groups, $SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$, Mahler's compactness criteria, Borel density theorem, Arithmetic lattices, Explaining the content of Borel-Harish-Chandra theorem and Margulis Arithmeticity theorem, Godment criteria for co-compactness of arithmetic lattices, Methods to produce arithmetic lattices.
T. N. Venkataramana (TNV) ICTS, Bengaluru	5 lectures	Title: A Proof of the Margulis Superrigidity Theorem Abstract: We shall review the proof of the Margulis superrigidity theorem following the paper by Margulis, Invent. Math. 1984, developing the necessary background. A brief outline is given below. Rank of a semisimple Lie group, Irreducible lattices, Uniform & non-uniform Lattices, Rank 1 vs Higher rank, Examples, Arithmeticity theorem with proof, Zariski dense subgroups, Furstenberg measure, Amenable subgroups, Existence of equivariant measurable maps, Induced representations, Rationality of equivariant measurable maps. Proof of the superrigidity theorem.
C. S. Rajan (CSR), Ashoka University	3 Lectures	Title: Harmonic superrigidity Abstract: The lectures will serve as an introduction to the rigidity results of Corlette and Gromov-Schoen.

Name of the Speaker	No. of Lectures	Abstract
Pranab Sardar (PS), IISER Mohali	1 lecture	Title: On some rigidity theorems in geometric group theory Abstract: The rigidity theorems of Mostow, Margulis and others for lattices in semisimple Lie groups inspired many such rigidity type results in geometry and topology. In this expository lecture, we will survey some such theorems related to geometric group theory.
Arghya Mondal	2 lectures	Title: Mostow rigidity Abstract: The Mostow Prasad Rigidity Theorem says if two finite volume hyperbolic manifolds of dimension greater than or equal to 3 are homotopy equivalent then they are isometric. We will give a complete proof of this statement for closed hyperbolic 3-manifolds. Then we will indicate how the proof can be extended in the general case.
Riddhi Shah (RS)	1 lecture	Title: The structure of automorphism groups and lattices in Lie groups Abstract: We study the structure of groups of automorphisms of a connected Lie group; namely, we identify certain conditions under which they are almost algebraic, and discuss some applications and examples (joint work with S.G. Dani - https://arxiv.org/abs/2504.18641). We explore the structure of lattices in a connected Lie group and discuss some properties of automorphisms which keep a lattice invariant (joint work with Rajdip Palit and Manoj B. Prajapati - Geometry, and Dynamics 17 (2023), 185-213 - https://doi.org/10.4171/GGD/672)

references

- [Dave Witte Morris - Arithmetic groups](#)
- [Venkataramana-rigidity-notes.pdf](#)
- [Raghunathan - Discrete subgroups of Lie groups](#)
- [Marc Bourdon \(Université de Lille\): "Quasi-isometric inv of continuous group \$L^p\$ cohomology & apps."](#) [Marc Bourdon - Quasi-isometric invariants of continuous group \$L^p\$ cohomology and applications](#)

- [Robert J. Zimmer - Ergodic Theory and Semisimple Groups \(1984, Birkhäuser\).pdf](#)
- [Werner Ballmann, Mikhael Gromov, Viktor Schroeder - Manifolds of Nonpositive Curvature \(1985, Birkhäuser\).pdf](#)

What is a rigidity theorem?

Mostow rigidity theorem

Quasi-isometric rigidity

Definition. quasi-isometric embedding

Let X, Y are metric spaces and $k \geq 0$. Then

$$f : X \rightarrow Y$$

is called a **k -quasi isometric embedding** if

$$-k + \frac{1}{k}d(x, x') \leq d(f(x), f(x')) \leq k + d(x, x')$$

for all $x, x' \in X$.

Examples:

- $$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R}^2 \\ t &\mapsto (t, \cos t) \end{aligned}$$

is a quasi embedd

- $$t \mapsto (t, t^2)$$

is **not** a quasi embedding

- $$t \mapsto (t, |t|)$$

is a quasi-isometry onto image



$$\mathbb{R}^m \cong_{\text{quasi}} \mathbb{R}^n \implies n = m$$



$$H^n \cong_{\text{quasi}} H^m \implies n = m$$

All our spaces are either manifolds... or graphs!

Consider a connected graph X and assign length 1 to each edge, which makes X a metric space

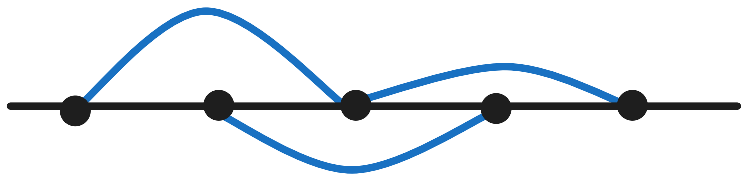
$$d(x, y) := \inf\{\text{length of paths from } x \text{ to } y\}$$

Now given a group Γ and $S \subset \Gamma$ we may consider its Caley graph $\text{Cay}(\Gamma, S)$ with vertices Γ and edges on (g, gs) for all $g \in \Gamma$ and $s \in S$.

Exercise

$\text{Cay}(\Gamma, S)$ is connected if $\Gamma = \langle S \rangle$.

Consider $\Gamma = \mathbb{Z}$ and $S = \{1, 2\}$ then the Caley graph is



(Milnor-Schwarz) Let X be a length metric space and $\Gamma \curvearrowright X$ acts by isometries properly such that X/Γ is compact then Γ is finitely generated and

$$\begin{aligned} \Gamma &\rightarrow X \\ g &\mapsto gx \end{aligned}$$

is a quasi-isometry

Corollaries:

- $\Gamma_1 < \Gamma$ (f.g.) and Γ/Γ_1 is finite then

$$\Gamma_1 \hookrightarrow \Gamma$$

is a quasi-isometry

- Let Γ be f.g. and $F \trianglelefteq \Gamma$ with $|F| < \infty$ then

$$\Gamma \rightarrow \Gamma/F$$

is a quasi-isometry.

- $$\text{Cay}(\Gamma, S_1) \cong_{\text{quasi}} \text{Cay}(\Gamma, S_2)$$

if $\langle S_1 \rangle = \langle S_2 \rangle = \Gamma$

Let X is a Riemannian symmetric space and $\Gamma < \text{Isom}(X)$ is a uniform lattice. Then

$$\begin{aligned} \Gamma &\rightarrow X \\ g &\mapsto gx \end{aligned}$$

is a quasi isometry.

Examples:

- $\mathbb{Z}^n \curvearrowright \mathbb{R}^n$
- $\pi_1(S_g) \curvearrowright H^2, g \geq 2$ then $\Gamma \cong_{\text{quasi}} H^2$

Definition. Virtual isomorphism of groups

Suppose Γ_1, Γ_2 are two f.g. groups. We say that Γ_1, Γ_2 are **virtually isomorphic** if there exists $\Gamma'_1 < \Gamma_1, \Gamma'_2 < \Gamma_2$ finite index and $F_1 \trianglelefteq \Gamma'_1, F_2 \trianglelefteq \Gamma'_2$ finite such that

$$\frac{\Gamma'_1}{F_1} \cong \frac{\Gamma'_2}{F_2}$$

Intuition

Suppose p is some group theoretic property and Γ is a group. We say that Γ satisfies p **virtually** if Γ has a finite index subgroup satisfying p .

For example being *virtually abelian*.

 Definition. Quasi-rigidity

A group Γ is called **quasi-rigid** if $\Gamma \cong_{\text{quasi}} \Gamma' \implies \Gamma$ is virtually isomorphic to Γ'

 Intuition

A group theoretic property p is called **quasi-rigid** if $\Gamma_1 \cong_{\text{quasi}} \Gamma_2$ and Γ_1 has property p implies there exists a Γ_3 with property p that is virtually isomorphic to Γ_2 .

 **(Gromov's polynomial growth theorem, Milnor, Wolf)** If Γ is quasi-isometric to a nilpotent group then Γ is virtually nilpotent.

 **(Stallings, Dunwoody et al)** If $\Gamma \cong_{\text{quasi}} F_n, n \geq 2$ then Γ is virtually free.

Example:

- $SL(2, \mathbb{Z}) = \mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$ is virtually free.

 Keywords

ends of groups, ends of metric spaces, chain of unbohnded components, throw balls, ...

 **(Gromov)** Γ is quasi-isometric to $\mathbb{Z}^n$ implies Γ is virtually isomorphic to $\mathbb{Z}^n$.

 $\Gamma \cong_{\text{quasi}} \mathbb{Z}$ implies Γ is virtually cyclic

The first step is:

Proposition: Γ is Gromov hyperbolic.

and then use:

 Γ infinite Gromov hyperbolic $\implies \Gamma$ has an infinite order element.

Let $\gamma \in \Gamma$ be an infinite order element.

The second step is:

 Let H hyperbolic group and $x \in H$ has infinite order. Then $\langle x \rangle \hookrightarrow H$ is a quasi-isometric embedding.

Hence $\mathbb{Z} \cong \langle \gamma \rangle \hookrightarrow_{\text{q-emb}} \Gamma \rightarrow_{\text{qi}} \mathbb{Z}$

- $\implies$ the composition is a qi

This implies

 Example

$\langle \gamma \rangle \hookrightarrow \Gamma$ is a qi

This implies

Proposition: $\Gamma / \langle \gamma \rangle$ is finite

proved by the fact that image of a quasi-isometry is coarsely surjective:

Exercise

If $f : X \rightarrow Y$ is a quasi-isometry then there exists $D \geq 0$ such that for all $y \in X \exists x \in X$ such that

$$d(f(x), y) \leq D$$

 **(Dyabina?) Solvability is not a quasi-rigid property.**

Examples of quasi-rigid properties:

- finite presentation
- Amenability
- hyperbolicity
- virtual cohomological dimension

Examples of quasi-rigid groups (quasi isometric implies virtually isomorphic):

- MCG of $S_g, g \geq 2$
- $\pi_1(S_g), g \geq 2$ (Gabai, Tukia, Cassen et al)
- π_1 of closed hyperbolic manifolds of $\dim \geq 3$ are quasi-rigid (Tukia, Sullivan)
- (Richard Schwartz) π_1 of finite volume hyperbolic ≥ 3 -manifolds are quasi-rigid
- Suppose X is an irreducible symmetric space of non-compact type of rank ≥ 2

-  **(Klienier, Leeb) Uniform lattices in $\text{Isom}(X)$ are quasi-rigid.**

-  **(Eskin) Non-uniform lattices of $\text{Isom}(X)$ are quasi-rigid.**

A non-example:

- (Burger-Mozhes) $F_2 \times F_2$ is not quasi-rigid

Problems

- Are polycyclic groups quasi-rigid.
- Are $SL(2, \mathbb{Z}) \times SL(n, \mathbb{Z}), n \geq 3$ quasi-rigid
- Are hyperbolic groups the form F_n

Keywords

max cohomological dim of MCG

Margulis' superrigidity Theorem

Let $\Gamma \leq G = \text{Isom}(X)$ acts on Y by isometry

$$\Gamma \rightarrow \text{Isom}(Y)$$

then this action extends to a action of G on Y .

 Question

Find similar theorem replacing Y by some other nice space

(Monad, Caprace, et al)

Generally this is false. There might be no nice action of Γ on Y .

Serre's property FA

 Definition. Serre's property FA

A group Γ is said to have property if for any isometric action of Γ on a tree T there is a **global fixed point**.

 **(Serre)** Suppose Γ is a countable group then Γ has **FA** $\iff \Gamma$ is fg, there is no surjective homomorphism $\Gamma \rightarrow \mathbb{Z}$ and Γ is not of the form $\Gamma_1 * \Gamma_2$.

We know $SL(2, \mathbb{Z})$ is an amalgam, but:

 **(Serre)** $SL(3, \mathbb{Z})$ has **FA**.

 **proof** Let $\Gamma = \langle \gamma_1, \dots, \gamma_n \rangle$ act on some tree.

Step 1: $\text{fix}(\gamma_i) \neq \emptyset$

- Proposition:** Given a nilpotent group $H \curvearrowright T$ acting on some tree
- either H has a global fixed point, or there exists a group homomorphism $H \rightarrow \mathbb{Z} \leq \text{Isom}(T)$ where $\mathbb{Z}$ acts by translation on the tree.
 - $[H, H] \curvearrowright T$ is trivial on the line

Step 2: $\text{fix}(\gamma_i) \cap \text{fix}(\gamma_j) \neq \emptyset$

Proposition: $\gamma_1 := I + E_{13}, \gamma_2, \gamma_3$ and

$$SL(3, \mathbb{Z}) = \langle \gamma_1, \gamma_2, \gamma_3 \rangle$$

Step 3:

Proposition: $H = \langle h_1, \dots, h_n \rangle \curvearrowright T$ and $\text{fix}(h_i) \neq \emptyset, \text{fix}(h_i) \cap \text{fix}(h_j) \neq \emptyset$ implies $\text{fix}(H) \neq \emptyset$

 Exercise

Step 2: $\text{fix}(\gamma_i) \cap \text{fix}(\gamma_j) \neq \emptyset$

$\implies \bigcap_i \text{fix}(\gamma_i) \neq \emptyset$

Proposition: Any finite group has FA.

Culler-Votgmann's criterion

Let $\Gamma = \langle \gamma_1, \dots, \gamma_n \rangle$ and $S = \{\gamma_1, \dots, \gamma_n\}$. We construct a graph $\Delta(S)$ with vertex set S and edges

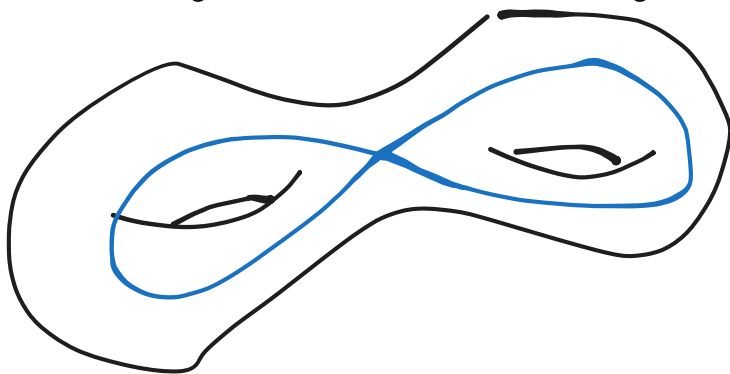
$$(\gamma_i, \gamma_j) \iff \exists \gamma_i^\pm \gamma_j^\pm \dots \text{ or } \gamma_j^\pm \dots \gamma_i^\pm$$

which commutes either with γ_i or γ_j . In particular if γ_i, γ_j commute.

Suppose $\Gamma \curvearrowright T$ $\Gamma = \langle S \rangle$, $\text{fix}(\gamma_i) \neq \emptyset$ for all $\gamma_i \in S$ and $\Delta(S)$ is complete.

• *corollary:*

- $SL(3, \mathbb{Z})$ has FA
- MCG for $g \geq 2$ has FA
 - "non intersecting Dehn twists commute, intersecting curves braid relation satisfies our hypothesis"



- $\text{Aut}(F_n), n \geq 3$ has FA

property T

Definition. property T

We say that a group G has **property T** if for any action of

$$G \rightarrow U(\mathcal{H})$$

for a Hilbert space $\mathcal{H}$ with an almost invariant vector

$$\forall \epsilon > 0, K \subset_c G, \exists v \in \mathcal{H}, \|v\| = 1 : \|\gamma v - v\| < \epsilon, \forall \gamma \in K$$

then it has a non-zero invariant vector.

Examples

- compact groups

Non-examples

- $\mathbb{Z}$
- F_n

(Alperin) Any locally compact group Γ has property T $\implies \Gamma$ has property FA

☰ Suppose Γ is a lattice in a semi-simple Lie group (of rank ≥ 2) with no rank 1 factor. Then Γ has property T.

☰ (Kostant) $Sp(n, 1)$ and any lattice in it has property T.

$SL(2, \mathbb{Z}) < SL(2, \mathbb{R})$ is a lattice in rank 1 semi-simple group which does not have property FA.

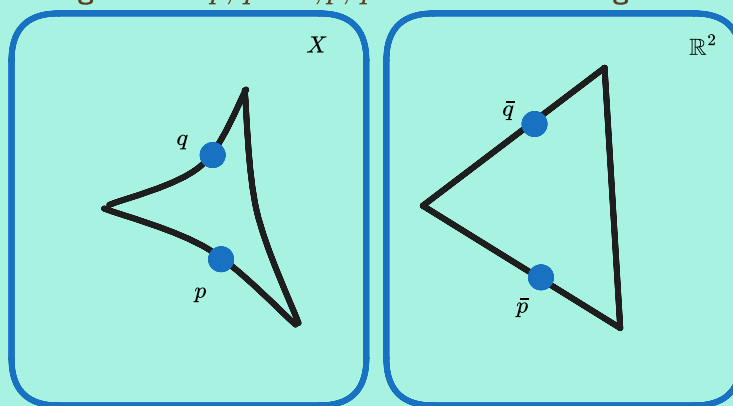
☰ (V-Gelfand-Graev) Lattices in $SO(n, 1), SU(n, 1)$ do not have property T.

CAT(0) spaces

Consider a geodesic metric space X that is every pair of points have a length minimizing curve between them

$$\forall x, y \in X, \exists \alpha : [0, a] \rightarrow X, \alpha(0) = x, \alpha(1) = y \\ d(\alpha(s), \alpha(t)) = |s - t|$$

📌 Definition. Consider a triangle in a metric space X and the corresponding triangle in $\mathbb{R}^2$ with standard metric with same side lengths. For $p, q \in X, \bar{p}, \bar{q} \in \mathbb{R}^2$ on the triangles the CAT(0)-inequality is



$$d(p, q) \leq d(\bar{p}, \bar{q})$$

📌 Definition. A geodesic metric space that satisfies the CAT(0) inequality is called CAT(0) space.

☰ X is a complete simply connected Riemannian manifold with sectional curvature $\leq 0 \implies X$ is CAT(0)

Examples:

- symmetric spaces of non-compact type
- CAT(0) cube complexes
- Euclidean buildings
- Salveth complex "RAAG"

📌 Definition. (Farb) FA_n

We say Γ has FA_n if for any isometric action of Γ on a CAT(0) complex X of dim n Γ has a global fixed point.

Thus FA is same as FA_n .

Proposition: $FA_n \implies FA_{n-1}$

☰ (Farb) $SL(n, \mathbb{Z})$ for $n \geq 3$ has FA_{n-2} (and this is the best).

☰ **(Helly)** Suppose $\{X_i\}$ is a finite collection of open convex sets in $\mathbb{R}^n$ such that

$$X_{i_1} \cap X_{i_2} \cap \cdots \cap X_{i_{n-1}} \neq \emptyset$$

$$\implies \bigcap_i X_i \neq \emptyset$$

☰ **(Brindson)** Suppose $\text{MCG}(S_g), g \geq 2$ is acting by isometries on complete CAT(0)-space of $\dim < g$. Then

$$\text{fix}(\text{MCG}) \neq \emptyset$$

☰ **(Bestma-Kapovich-Kleiner)** Suppose X is a symmetric space of non-compact type and $\Gamma < \text{Isom}(X)$ is a lattice. Suppose Γ is acting properly discontinuously on a contractible n -manifold Y . Then $n \geq \dim(Y)$.

Pralay Chatterjee - Introduction to Lattices

📖 Outline

- Generalities on lattices in Lie groups
- $SL(n, \mathbb{Z})$ is a lattice in $SL(n, \mathbb{R})$
- Borel density theorem
- state Borel-Harish-Chandra theorem
- Godment criteria for co-compactness of arithmetic lattices
- state Margulis Arithmeticity theorem for lattices in higher rank Lie groups
- Methods to produce arithmetic lattices

All groups will be second countable and locally compact. But mostly they will be Lie groups.

☰ Every locally compact group G posses a **left Haar measure** and it is unique up to a multiplicative constant.

📌 Definition. Modular function of a locally compact group

Let μ be a left invariant Haar measure. And

$$\begin{aligned} \mathfrak{B}_G &\rightarrow \mathbb{R} \\ A &\mapsto \mu(Ag) \end{aligned}$$

Then we have the modular function

$$\begin{aligned} \Delta : G &\rightarrow \mathbb{R}_{>0} \\ \mu(Ag) &:= \Delta(g^{-1})\mu(A) \end{aligned}$$

or all $A \in \mathfrak{B}_G$.

It is a continuous group homomorphism.

If G is a Lie group then we can choose a top form $\omega \in \Omega^{\dim G}(G)$. This gives a left invariant Haar measure.

Then it is easy to calculate the modular function:

Let $\mathfrak{g}$ be the Lie algebra of G . Then we have the adjoint representation

$$\begin{aligned}\mathrm{Ad} : G &\rightarrow GL(\mathfrak{g}) \\ \mathfrak{g} &\mapsto \mathrm{d}_i c_g\end{aligned}$$

and

$$\Delta(g) = |\det \mathrm{Ad}(g)|$$

📌 **Definition.** If $\Delta(G) = \{1\}$ then G is called **unimodular**.

Following groups are unimodular:

- compact groups
- G such that $G = [G, G]$
- semi-simple G
- connected nilpotent Lie group

However,

- solvable groups are **not** in general unimodular

All our measures are *always* regular.

📌 **Definition.** ρ -semi invariant measures

Let

- $\rho : G \rightarrow \mathbb{R}_{>0}$ be a continuous character
- $(X, \mathfrak{M}, \mu)$ be a measure space
- $G \curvearrowright X$ be measurable group action

then μ is said to be **ρ -semi invariant** if $\mu(gA) = \rho(g)\mu(A)$ for all $g \in G, A \in \mathfrak{M}$.

Let $H < G$ be a closed subgroup of a Lie group G .

Then G/H is a manifold and

$$G \curvearrowright G/H$$

📖 Let $H < G$ be a closed subgroup of a Lie group(?or locally compact?) G . Let $\rho : G \rightarrow \mathbb{R}_{>0}$ be a continuous character. Then $\exists$ ρ -semi invariant measure on G/H

$$\Longleftrightarrow$$

$\rho(h) = \Delta_G(h)\Delta_H(h^{-1})$ for all $h \in H$. Moreover, (given fixed ρ) if one of the above holds, any two such measures will be positive multiples of each other.

💡 Reference

#book [M S Raghunathan - Discrete Subgroups of Lie Groups - Springer \(1972\).pdf](#) #course/mini [Discrete subgroups of Lie groups - YouTube](#)

- (*corollary*) If H is unimodular G/H admits a Δ -semi invariant measure

- (corollary) If $\Gamma < G$ is discrete, then G/H admits a Δ_G -semi invariant measure

Proposition: Suppose G is a Lie group and $H < G$ and H is unimodular. Suppose G/H has a G -invariant measure such that $\mu(G/H) < \infty$ then G is unimodular.



$$\mu\left(g\frac{G}{H}\right) = \Delta(g)\mu\left(\frac{G}{H}\right)$$

thus $\Delta(g) = 1$

💡 $\Delta_G(H) = 1 \implies H \subset N := \ker \Delta_G$ and then we have

$$\pi : \frac{G}{H} \rightarrow \frac{G}{N}$$

- $\pi^*(\mu)$ is a measure on G/N
- $\implies \pi^*(\mu)$ is a finite Haar measure on G/N
- $\implies G/\ker \Delta_G$ is compact
- thus $\Delta_G(G) = 1$

🔑 **Definition.** Let G be a Lie group and $\Gamma < G$ is a discrete subgroup then Γ is a **lattice** if G/Γ admits a G -invariant measure such that $\mu(G/\Gamma) < \infty$.

🔑 **Definition.** Let $\Gamma < G$ is discrete. Then a Borel set $F \subset G$ is called a **fundamental domain** for Γ if $G = F\Gamma$ and $F \cap F\gamma = \emptyset$ if $\gamma \neq i_G$.

📋 Let $\Gamma < G$ is any discrete subgroup then it admits a fundamental domain .

💡 Let $U \subset G$ be open such that $U^{-1}U \cap \Gamma = \{i_G\}$

- G is second countable $\implies$

$$Ug_nU = G$$

$$F := \bigcup_{n=1}^{\infty} \left(g_nU - \bigcup_{i < n} g_iU\Gamma \right)$$

Proposition: If F is a fundamental domain, $\gamma_i \in \Gamma$ and $F = \bigcup_{i \geq 1} A_i$ disjoint. Then

$$\bigcup_{i \geq 1} A_i \gamma_i$$

is another fundamental domain. Any fundamental domain can be obtained from another by such a process.

Proposition: If $C \subset G$ such that

$$\frac{C}{G} = \frac{G}{\Gamma} \iff C\Gamma = G$$

then there exists a fundamental domain $F \subset C$.

Proposition: If $B \subset G$ is such that $B \rightarrow \frac{G}{\Gamma}$ is injective then there exists a fundamental domain $F \supset B$.

 **Definition.** Let ν be the right invariant Haar measure on G and F be a fundamental domain for Γ . Then

$$\nu_{G/\Gamma}(A) := \nu_G(\pi^{-1}(A) \cap F)$$

for all Borel $A \subset G/\Gamma$ and $\pi : G \rightarrow G/\Gamma$.

Exercise

Check that $\nu_{G/\Gamma}$ is independent of the fundamental domain chosen.

Use

Proposition: If F is a fundamental domain, $\gamma_i \in \Gamma$ and $F = \bigcup_{i \geq 1} A_i$ disjoint. Then

$$\bigcup_{i \geq 1} A_i \gamma_i$$

is another fundamental domain. Any fundamental domain can be obtained from another by such a process.

This measure is

- $B \subset G$ then $\nu_G(B) \geq \nu_{G/\Gamma}(\pi(B))$ and equal when π is injective on B
- **regular:** for compact $K \subset G/\Gamma$ choose $C \subset G$ compact such that $\pi(C) = K$ then

$$\infty > \nu_G(C) \geq \nu_{G/\Gamma}(K)$$

- Δ -semi invariant



1.12. Theorem. Let G be a locally compact group satisfying the second axiom of countability and $\Gamma \subset G$ a lattice. Let $\{x_n\}_{1 \leq n < \infty}$ be a sequence in G . Let $\pi : G \rightarrow G/\Gamma$ be the natural map. Then the sequence $\pi(x_n)$ has no convergent subsequence in G/Γ if and only if there exists a sequence $\{\gamma_n\}_{1 \leq n < \infty}$ in Γ such that $\gamma_n \neq e$ for any n and $x_n \gamma_n x_n^{-1}$ converges to e as n tends to ∞ .

$SL(n, \mathbb{Z})$ is a lattice in $SL(n, \mathbb{R})$



$SL(n, \mathbb{Z})$ is a lattice in $SL(n, \mathbb{R})$

 **proof:** To get a Borel subset C such that

$$SL(n, \mathbb{R}) = CT$$

such that $\mu(C) < \infty$.

Proposition: $SL(n, \mathbb{R}) = KA_{\sqrt{2}/3}N_{1/2}\Gamma$



Let G be a Lie group and $S, T \leq G$ be closed, $T \cap S$ compact with $S \times T \rightarrow G$ product map open and surjective. Then

$$\int f \delta \mu_G = \int_{S \times T} f(st) \frac{\Delta_T(t)}{\Delta_G(t)} \delta \mu_S(s) \delta \mu_H(t)$$

Riddhi Shah - The structure of automorphism groups and lattices in Lie groups

✦ Abstract

We study the structure of groups of automorphisms of a connected Lie group; namely, we identify certain conditions under which they are almost algebraic, and discuss some applications and examples (joint work with S.G. Dani - <https://arxiv.org/abs/2504.18641>). We explore the structure of lattices in a connected Lie group and discuss some properties of automorphisms which keep a lattice invariant (joint work with Rajdip Palit and Manoj B. Prajapati - Geometry, and Dynamics 17 (2023), 185-213 - <https://doi.org/10.4171/GGD/672>)

$$\text{Aut}(G) \rightarrow \text{AutLieAlg}(\mathfrak{g}) \subset GL(\mathfrak{g})$$

is one to one

$$\exp(d\tau(X)) = \tau(\exp X)$$

📌 **Definition.** $\text{Aut}(G)$ is **almost algebraic** if it is open subgroup finite index in an algebraic group.

📖 **(Wigner, 1976)** $\text{Aut}(G)_0$ is almost algebraic.

📖 **(Dani 1992)** Let C be the max cpt connected central (max central tori) subgroup of G . Then $\text{Aut}(G)$ is almost algebraic if C is trivial.

	Aut
T^n	$GL(n, \mathbb{Z})$
$\mathbb{R}^n$	$GL(n, \mathbb{R})?$
simply connected G	$\text{AutLieAlg}(\mathfrak{g})?$ Zariski closed? algebraic...
G/C as in 📖 (Dani 1992) Let C be the max cpt connected central (max central tori) subgroup of G. Then $\text{Aut}(G)$ is almost algebraic if C is trivial.	$\text{Aut}(G/C)$ is almost algebraic
N_3	
$SL(2, \mathbb{R}) \backslash \text{semi} G$	almost alg
$SO(2, \mathbb{R}) \backslash \text{semi} G$	not alg!

....

C S Rajan - Harmonic super-rigidity of Corlette and Gromov-Schoen

III. Vector bundle valued harmonic forms

(1) Betti numbers of locally symmetric spaces, [24, 25, 28, 31].

In 1960, Matsushima returned to his Alma Mater, Osaka University, to fill the Chair of Algebra vacated by his teacher, Professor Shoda. It was in Osaka that he wrote a series of important papers on cohomology of locally symmetric spaces.

According to Kodaira's theory of deformations, the space of infinitesimal deformations of a pseudogroup structure is given in terms of the first cohomology with coefficients in the sheaf of germs of infinitesimal automorphisms. On the other hand, this cohomology group can be represented by harmonic forms by the Hodge theory. Using Bochner's technique of proving vanishing theorems, Calabi and Vesentini [1] obtained vanishing of certain cohomology groups $H^q(M, \Theta)$ for compact quotients M of symmetric bounded domains and their sheaves Θ of germs of holomorphic vector fields. Using the same idea, Weil

[1] proved the infinitesimal rigidity for uniform discrete subgroups of semisimple Lie groups.

In [24], Matsushima proved that if M is a compact quotient of a symmetric bounded domain X none of whose irreducible factors is equivalent to the unit ball in $\mathbb{C}^n$, then the first cohomology $H^1(M, \mathbb{R})$ vanishes. This brought out in striking contrast the difference between the classical 1-dimensional case and the higher dimensional case. As in the work of Calabi-Vesentini and Weil, positivity of a certain quadratic form defined by the curvature is the essential point of the proof.

In [25], Matsushima considered higher dimensional cohomology $H^q(M; \mathbb{R})$. If X^* is a symmetric space of compact type, then $H^q(X^*; \mathbb{R})$ or the space of harmonic q -forms on X^* is given, according to E. Cartan, by invariant parallel q -forms on X^* . These q -forms induce invariant parallel q -forms on the non-compact dual X which, in turn, induce parallel (and hence, harmonic) q -forms on the quotient M . Hence, $\dim H^q(M; \mathbb{R}) \geq \dim H^q(X^*; \mathbb{R})$ in general. Considering again positivity of a similar quadratic form, Matsushima determined the range of q for which the equality $H^q(M; \mathbb{R}) = H^q(X^*; \mathbb{R})$ holds for each symmetric bounded domain X . Kaneyuki and Nagano [1], [2] completed Matsushima's program by treating the remaining symmetric spaces X of noncompact type. Matsushima's lecture at the Bombay Colloquium [28] is a survey of his and their results on Betti numbers of locally symmetric spaces.

Matsushima's paper [31] which relates Betti numbers of locally symmetric spaces to unitary representations is of growing importance. Let $X = G/K$ be a symmetric space of noncompact type and $M = \Gamma \backslash X$ a compact locally symmetric space. In [31] he gave a formula for the Betti numbers of M in terms of the multiplicities of certain unitary representations of G in the right regular representation of G on $L^2(\Gamma \backslash G)$. Hotta and Wallach [1] investigated existence and non-existence of unitary representations appearing in the Matsushima formula and explain vanishing theorems in Matsushima's earlier papers [24], [25].

We consider families of complex manifold:

$$\begin{array}{ccc} \mathcal{X} & \text{total space} & \\ \downarrow \pi & & \\ S & \text{parameterizing space} & \end{array}$$

the fibres $\mathcal{X}_s := \pi^{-1}(s)$ is space with *extra structure*.

Intuition

A space $X_0 \cong \mathcal{X}_{s_0}$ is **locally rigid** if for every s in a sufficiently small nbd of $s_0 \in S$

$$\mathcal{X}_s \cong \mathcal{X}_{s_0} \cong X_0$$

This is *hard* to verify. Instead, we replace this with infinitesimal or cohomological condition in terms of X_0 .

Ehresmann’s theorem for smooth manifolds

 **(Ehresmann)** Let $\mathcal{X}, S$ are manifolds and

$$\begin{array}{c} \mathcal{X} \\ \downarrow \pi \\ S \end{array}$$

is be a proper, smooth submersion. Then locally on S , π is a product

$$\begin{array}{ccc} \pi^{-1}(U) = \mathcal{X}_U & \rightarrow & X_{s_0} \times U \\ \downarrow & & \downarrow \\ U & = & U \end{array}$$

 *proof* by implicit function theorem we have

- the vector field

$$\frac{\partial}{\partial t} \text{ on } U$$

and pull back by π_* and patch using partition of unity

- ...

[1]

for complex manifolds

The partition of unity argument fails.

[2]

1-cocycle of holomorphic vector fields on the fibre X_0

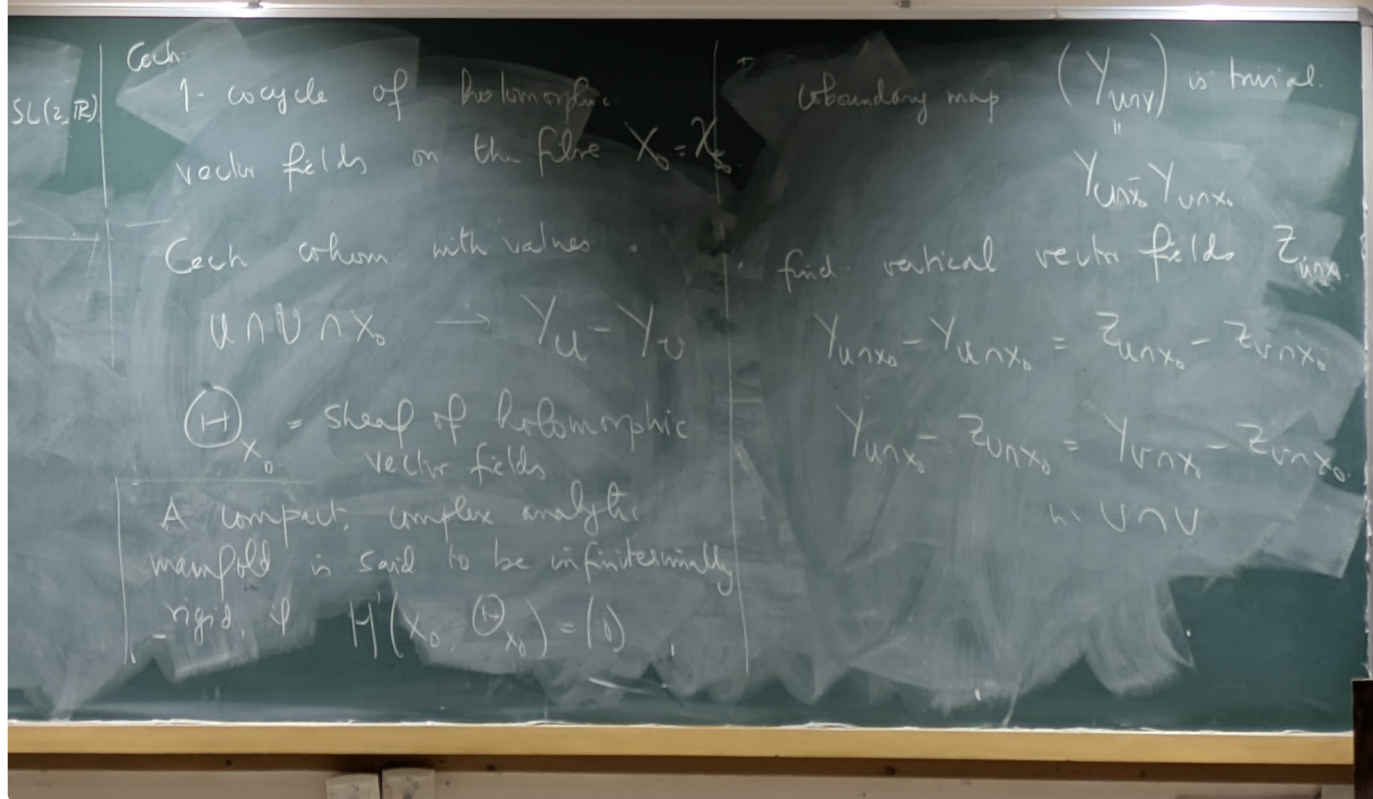
Cech cohomology with values

...

 **Definition. Infinitesimal rigidity**

A compact, complex manifold is said to be **infinitesimally rigid** if

$$H^1(X_0, \mathcal{O}_{X_0}) = (0)$$



Calabi-Visentini rigidity theorem

Calabi-Vesentini rigidity theorem:

X compact, locally hermitian symmetric, (simple), not of complex dimension 1

Then X is infinitesimally rigid

Selberg, Calabi

local rigidity: lattice in $SL(n, \mathbb{R})$, $n \geq 3$

orthogonal

Wad. local rigidity for lattices in simple Lie groups not isogenous to $SL(2, \mathbb{R})$ (Matsushima)

$p_0: \Gamma \subset \text{lattice in } G$

$p_t: \Gamma \rightarrow G$

$|t| < \varepsilon$

$$p_t(\gamma \gamma') = p_t(\gamma) \cdot p_t(\gamma')$$

Let X be compact, locally Hermitian symmetric spaces of $\dim > 1$ then X is infinitesimally rigid.

$\mathbb{Z}, \mathbb{R}$

local rigidity:

there exists $g_t \in G$, such that

$$p_t = g_t p_0 g_t^{-1}$$

1-cycle valued in $\mathfrak{g} = \text{Lie}(G)$ of Γ

Tangent space at p_0

$$Z_{p_0}^1(\Gamma, \mathfrak{g}) \rightarrow 1\text{-cycles}$$

Γ is finitely presented

S set of generators for Γ

$$G^S = \text{Maps}(S, G)$$

$$s \in S \quad p_t(s) \in G$$

$$p_t(s) \in G^{|S|}$$

$$s_1, s_2 = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$$

finitely many relations

$$\text{Rep}(\Gamma, G) \subseteq G^S$$

cut out by the relation

$$\Gamma \rightarrow \mathfrak{g}$$

$$c(x) = \left. \frac{d}{dt} \right|_{t=0} (p_t(x) p_0(x)^{-1})$$

$$p_t(x x') = p_t(x) \cdot p_t(x')$$

$$c(x x') = ?$$

$$p_t(x) (p_t(x') p_0(x')^{-1}) p_0(x)^{-1}$$

$$= p_t(x) (p_t(x') p_0(x')^{-1}) p_t(x)^{-1} p_t(x) p_0(x)^{-1}$$

$$t=0 \quad c(x x') = c(x) + p_0(x) c(x') p_0(x)^{-1}$$

$$\in Z_{p_0}^1(\Gamma, \mathfrak{g})$$

1-cocchain $\exists a \in G$

$$b(x) = p_0(x) a p_0(x)^{-1} = a$$

cocompact lattices

Wells then: Non $SL_2(\mathbb{R})$, simple gps

$$H_{p_0}^1(\Gamma, \mathfrak{g}) = (0)$$

infinitely rigid

$$p_0 \Gamma \subseteq G \text{ cocompact lattice}$$

— local systems

— Hodge theory

— Bochner-Weitzenböck

↓

curvature calculations
(relative Lie alg cohomology)

$$(X, x_0), \quad \Gamma = \pi_1(X, x_0)$$

universal cover $(\tilde{X}, \tilde{x}_0)$

$$\downarrow$$

$$(X, x_0)$$

$$\Gamma \xrightarrow{P} GL(V)$$

discrete topology on V .

$$\mathcal{L}_P = \tilde{X} \times_{\pi_1(X, x_0)} V_P$$

$$\downarrow$$

$$X \quad X$$

vector bundle with a flat connection

$$\mathcal{L}_P \otimes_{\mathbb{C}} \mathcal{O}_X \quad (\text{locally free sheaf})$$

$$\downarrow$$

$$X$$

$\mathcal{L}$

X

Seri. An. Ia D

$$\Delta_d = \Delta_D + \text{linear term}$$

(curvature term)

$$\Delta_d \geq 0, \quad \Delta_D \geq 0$$

$$(\Delta_d w, w) = (\Delta_D w, w) + (R(w), w)$$

≥ 0

1. orbifold $\exists a \in G$

$$b(x) = p_0(x) a p_0(x)^{-1} = a$$

cocompact
lattices

Wels then Non $SL_2(\mathbb{R})$, simple ops

$$H'_0(\Gamma, \mathfrak{g}) = (0) \quad \text{infinitely rigid}$$

$$p_0 \Gamma \subseteq G \quad \text{cocompact lattice}$$

$$\{\rho : \Gamma \rightarrow GL(V_\rho)\}$$

? $\leftrightarrow$ local systems (discrete topology on V_ρ) $\leftrightarrow$ vector bundles with flat connection (standard topology on V_ρ)

Bochner's method

Let M be a Riemannian manifold and

$$\begin{array}{c} (E, \langle, \rangle, D) \\ \downarrow \\ (M, g) \\ X \in \text{Vec}(M) \end{array}$$

- D is a connection, so $s \in \mathbf{\Gamma}(U, E)$ then

$$D_X s \in \mathbf{\Gamma}(U_1, E)$$

- D is *compatible with M* that is

$$X \langle s, t \rangle = \langle D_X s, t \rangle + \langle s, D_X t \rangle$$

- E also has other natural operators
 - if E is associated to a local system $\mathcal{L}_\rho \rightarrow M$ then

$$\begin{array}{c} E_\rho = \mathcal{O}_M \otimes \mathcal{L}_\rho \\ \downarrow \\ M \end{array}$$

then it has a connection

$$D_{\mathcal{L}} : E_\rho \rightarrow T^*M \otimes E_\rho$$

- (Weitzenböck) E is the vector bundle of p -forms $p \geq 1$

$$\mathbf{A}^p T^*M \overset{\text{d}^p}{\longrightarrow} \mathbf{A}^{p+1} T^*M$$

- Laplacian*

$$\begin{array}{c} E \overset{D}{\longrightarrow} \Omega^1 \otimes E \overset{D}{\longrightarrow} \Omega^2 \otimes E \rightarrow \dots \\ \Delta_D = D^*D + DD^* \end{array}$$

- Bochner-Laplacian*

$$\Delta := DD^*$$

Lie algebra cohomology

$$\mathfrak{g} \rightarrow \text{End}(M)$$

we define

$$H^*(\mathfrak{g}, M)$$

which has a usual co-chain model

$$\{f : \Lambda^i(\mathfrak{g}) \rightarrow M\} =: C^i(\mathfrak{g}, M) \overset{d}{\rightarrow} C^{i+1}(\mathfrak{g}, M)$$

where

$$\text{d}f(x_1, \dots, x_{i+1}) = \sum (-1)^{i+1} x_i f(x_1, \dots, \hat{x}_i, \dots, x) + \sum$$

where

$$H^i=\frac{\ker}{\mathrm{im}}$$

Relative Lie algebra cohom

$$K\leq G, \mathfrak{k}\leq \mathfrak{g}$$

Bernstein

$$R(\mathfrak{g},K):=\text{distributions on }G\text{ supp in }K$$

where K is compact and the algebra is under convolution, this is the *Hecke algebra at infinity*

$$\text{Out }(\mathfrak{g},K)\text{-modules}\leftrightarrow R(\mathfrak{g},K)\text{-modules}$$

$$C^i(\mathfrak{g},K,M):=K\text{-equivariant maps }\Lambda^i(\mathfrak{g}/\mathfrak{k},M)$$

$$f(kx_1,\dots,kx_i)=\rho(k)^{-1}f(x_1,\dots,x_i)$$

semi-simple $G\geq K$ maximal compact with

$$\rho:G\rightarrow \mathrm{Aut}(V_\rho)$$

$$M=V_\rho \text{ finite dim then}$$

$$H^*(\Gamma,\rho)\cong H^*\left(\mathfrak{g},K,\mathcal{C}^\infty\left(\frac{G}{\Gamma}\right)\otimes V_\rho\right)$$

$$\begin{array}{ccc} & G & \\ \swarrow & & \searrow \\ G/K & & G/\Gamma \end{array}$$

$$(\Delta^p\eta)(x_i)$$

The chronology of rigidity begins with the theorem of A. Selberg ([16]) that a discrete co-compact subgroup Γ of $SL(n, \mathbf{R})$ cannot be continuously deformed except trivially, that is, by inner automorphisms of $SL(n, \mathbf{R})$, if $n > 2$; Selberg's proof rested on showing that the trace of elements in Γ are preserved under deformations of Γ . Selberg's method applied to the other classical groups of rank greater than 1. At about the same time, E. Calabi and E. Vesentini proved the rigidity of complex structure under infinitesimal deformations of compact quotients of bounded symmetric domains ([3b]), and later Calabi proved the metric analogue for compact hyperbolic n -space forms for $n > 2$ ([3a]). Thereupon A. Weil ([21]) generalized Selberg's and Calabi's results to semi-simple groups having no compact or 3 dimensional simple factors. Weil's proof deduces the rigidity of Γ in G under deformations from the vanishing of the cohomology group $H^1(\Gamma, \dot{G})$ where $\dot{G}$ is the Lie algebra of G regarded as a Γ -module under the adjoint representation. In the case of arithmetic subgroups, which are lattices (that is, Γ is discrete in G and G/Γ has finite Haar measure) but not generally co-compact, the rigidity under deformations was proved independently by A. Borel (unpublished) in the case of $\mathbf{Q}$ -simple groups of $\mathbf{Q}$ -rank at least two, and H. Garland (in the split case), and by M. S. Raghunathan in the remaining $\mathbf{Q}$ -rank one cases, again by showing that $H^1(\Gamma, \dot{G}) = 0$.

3.1. Local and global rigidity of homomorphisms into finite dimensional groups. The earliest work on local rigidity in the context of Definition 1.1 was contained in series of works by Calabi–Vesentini, Selberg, Calabi and Weil, which resulted in the following:

THEOREM 3.1. *Let G be a semisimple Lie group and assume that G is not locally isomorphic to $SL(2, \mathbb{R})$. Let $\Gamma < G$ be an irreducible cocompact lattice, then the defining embedding of Γ in G is locally rigid.*

REMARKS. (1) If $G = SL(2, \mathbb{R})$ the theorem is false and there is a large, well studied space of deformation of Γ in G , known as the Teichmüller space.
 (2) There is an analogue of this theorem for lattices that are not cocompact. This result was proven later and has a more complicated history which we omit here. In this case it is also necessary to exclude G locally isomorphic to $SL(2, \mathbb{C})$.

This theorem was originally proven in special cases by Calabi, Calabi–Vesentini and Selberg. In particular, Selberg gives a proof for cocompact lattices in $SL(n, \mathbb{R})$ for $n \geq 3$ in [S], Calabi–Vesentini give a proof when the associated symmetric space $X = G/K$ is Kähler in [CV] and Calabi gives a proof for $G = SO(1, n)$ where $n \geq 3$ in [C]. Shortly afterwards, Weil gave a complete proof of Theorem 3.1 in [We1; We2].

In all of the original proofs, the first step was to show that any perturbation of Γ was discrete and therefore a cocompact lattice. This is shown in special cases in [C; CV; S] and proven in a somewhat broader context than Theorem 3.1 in [W1].

The different proofs of cases of Theorem 3.1 are also interesting in that there are two fundamentally different sets of techniques employed and this dichotomy continues to play a role in the history of rigidity. Selberg’s proof essentially combines algebraic facts with a study of the dynamics of iterates of matrices. He makes systematic use of the existence of singular directions, or Weyl chamber walls, in maximal diagonalizable subgroups of $SL(n, \mathbb{R})$. Exploiting these singular directions is essential to much later work on rigidity, both of lattices in higher rank groups and of actions of abelian groups. It seems possible to generalize Selberg’s proof to the case of G an $\mathbb{R}$ -split semisimple Lie group with

rank at least 2. Selberg's proof, which depended on asymptotics at infinity of iterates of matrices, inspired Mostow's explicit use of boundaries in his proof of strong rigidity [Mo2]. Mostow's work in turn provided inspiration for the use of boundaries in later work of Margulis, Zimmer and others on rigidity properties of higher rank groups.

The proofs of Calabi, Calabi–Vesentini and Weil involve studying variations of geometric structures on the associated locally symmetric space. The techniques are analytic and use a variational argument to show that all variations of the geometric structure are trivial. This work is a precursor to much work in geometric analysis studying variations of geometric structures and also informs later work on proving rigidity/vanishing of harmonic forms and maps. The dichotomy between approaches based on algebra/dynamics and approaches that are in the spirit of geometric analysis continues through much of the history of rigidity and the history of local rigidity of group actions in particular.

Shortly after completing this work, Weil discovered a new criterion for local rigidity [We3]. In the context of Theorem 3.1, this allows one to avoid the step of showing that a perturbation of Γ remains discrete. In addition, this result opened the way for understanding local rigidity of more general representations of discrete groups.

THEOREM 3.2. *Let Γ be a finitely generated group, G a Lie group and $\pi : \Gamma \rightarrow G$ a homomorphism. Then π is locally rigid if $H^1(\Gamma, \mathfrak{g}) = 0$. Here $\mathfrak{g}$ is the Lie algebra of G and Γ acts on $\mathfrak{g}$ by $\text{Ad}_G \circ \pi$.*

Weil's proof of this result uses only the implicit function theorem and elementary properties of the Lie group exponential map. The same theorem is true if G is an algebraic group over any local field of characteristic zero. In [We3], Weil remarks that if $\Gamma < G$ is a cocompact lattice and G satisfies the hypothesis of Theorem 3.1, then the vanishing of $H^1(\Gamma, \mathfrak{g})$ can be deduced from the computations in [We2]. The vanishing of $H^1(\Gamma, \mathfrak{g})$ is proven explicitly by Matsushima and Murakami in [MM].

Motivated by Weil's work and other work of Matsushima, conditions for vanishing of $H^1(\Gamma, \mathfrak{g})$ were then studied by many authors. See particularly [MM] and [Rg1]. The results in these papers imply local rigidity of many linear representations of lattices.

To close this section, I will briefly discuss some subsequent developments concerning rigidity of lattices in Lie groups that motivated the study of both local and global rigidity of group actions.

The first remarkable result in this direction is Mostow's rigidity theorem, see [Mo1] and references there. Given G as in Theorem 3.1, and two irreducible

cocompact lattices Γ_1 and Γ_2 in G , Mostow proves that any isomorphism from Γ_1 to Γ_2 extends to an isomorphism of G with itself. Combined with the prin-

principal theorem of [We1] which shows that a perturbation of a lattice is again a lattice, this gives a remarkable and different proof of Theorem 3.1, and Mostow was motivated by the desire for a “more geometric understanding” of Theorem 3.1 [Mo1]. Mostow’s theorem is in fact a good deal stronger, and controls not only homomorphisms $\Gamma \rightarrow G$ near the defining homomorphism, but any homomorphism into any other simple Lie group G' where the image is lattice. As mentioned above, Mostow’s approach was partially inspired by Selberg’s proof of certain cases of Theorem 3.1, [Mo2]. A key step in Mostow’s proof is the construction of a continuous map between the geometric boundaries of the symmetric spaces associated to G and G' . Boundary maps continue to play a key role in many developments in rigidity theory. A new proof of Mostow rigidity, at least for G_i of real rank one, was provided by Besson, Courtois and Gallot. Their approach is quite different and has had many other applications concerning rigidity in geometry and dynamics; see [BCG; CF], for example.

The next remarkable result in this direction is Margulis’ superrigidity theorem. Margulis proved this theorem as a tool to prove arithmeticity of irreducible uniform lattices in groups of real rank at least 2. For irreducible lattices in semisimple Lie groups of real rank at least 2, the superrigidity theorems classifies all finite dimensional linear representations. Margulis’ theorem holds for irreducible lattices in semisimple Lie groups of real rank at least two. Given a lattice $\Gamma < G$ where G is simply connected, one precise statement of some of Margulis results is to say that any linear representation σ of Γ *almost extends* to a linear representation of G . By this we mean that there is a linear representation $\tilde{\sigma}$ of G and a bounded image representation $\bar{\sigma}$ of Γ such that $\sigma(\gamma) = \tilde{\sigma}(\gamma)\bar{\sigma}(\gamma)$ for all γ in G . Margulis’ theorems also give an essentially complete description of the representations $\bar{\sigma}$, up to some issues concerning finite image representations. The proof here is partially inspired by Mostow’s work: a key step is the construction of a measurable “boundary map”. However the methods for producing the boundary map in this case are very dynamical. Margulis’ original proof used Oseledec Multiplicative Ergodic Theorem. Later proofs were given by both Furstenberg and Margulis using the theory of group boundaries as developed by Furstenberg from his study of random walks on groups [Fu1; Fu2]. Furstenberg’s probabilistic version of boundary theory has had a profound influence on many subsequent developments in rigidity theory. For more discussion of Margulis’ superrigidity theorem, see [Ma2; Ma3; Ma4].

A main impetus for studying rigidity of group actions on manifolds came from Zimmer’s theorem on superrigidity for cocycles. This theorem and its proof were strongly motivated by Margulis’ work. In fact, Margulis’ theorem is Zimmer’s theorem for a certain cocycle $\alpha : G \times G / \Gamma \rightarrow \Gamma$ and the proof of Zimmer’s theorem is quite similar to the proof of Margulis’. In order to avoid

Zimmer's theorem is quite similar to the proof of Margulis. In order to avoid

Margulis super-rigidity

Following: [Venkataramana-rigidity-notes.pdf](#)

Tldr

super-rigidity $\implies$ Arithmeticity

Let $G = PSL(n, \mathbb{R}), n \geq 3$.

We want to show Γ is arithmetic from this theorem:

 **(super-rigidity)** Let $L = \mathbb{R}, \mathbb{C}, K \underbrace{\geq}_{\text{finite}} \mathbb{Q}_p$ (locally compact fields of char 0) and H be a algebraic group of adjoint type. Let G is non-compact simple Lie group of adjoint type, Γ be a fg lattice (so irred) in G . A rep

$$\rho : \Gamma \rightarrow H(L)$$

continuous with $\rho(\Gamma)$ is non-compact. Then ρ extends to a continuous map

$$G \rightarrow H(L)$$

Marc Bourdon - Lie groups and quasi-isometries

Abstract

In geometric group theory, one studies groups as geometric objects, and tries to determine whether an algebraic property is a geometric one. In this setting, the natural maps between groups are the so-called quasi-isometries (QI). A class of groups is said to be QI-rigid if every finitely generated group that is QI to a group in the class, is virtually isomorphic to (another) group in the class.

Geometric group theory started in the 80's when Gromov proved that the class of nilpotent groups is QI-rigid. In the 90's the subject focused on semisimple groups; it culminated when a combination of works allowed to establish that the class of lattices in a given semisimple group is QI-rigid.

Since then, geometric group theory has developed several interactions with other fields of mathematics. It has been used to solve some old open problems (e.g. in 3-manifold topology).

A major problem in geometric group theory is to classify connected Lie groups up to QI. One knows that every simply connected Lie group is QI to a completely solvable Lie group, i.e. to a closed subgroup of the upper triangular real matrix group. In 2018 Cornuier conjectured that two QI completely solvable groups must be isomorphic. This is currently open, even in the smaller class of nilpotent Lie groups.

During the talks, I plan to introduce some QI-invariants for Lie groups, and illustrate them with examples. These invariants include the growth rate, the rank, and the L^p -cohomology. The examples of groups that will be discussed include the nilpotent groups, the Heintze groups, the abelian-by-abelian solvable Lie groups. Some known results about their QI classification will also be presented.

Let G be a finitely generated or connected Lie group.

We geometrize G :

- If G is fg let S be a finite set of generators

$$|g|_S := \min\{k \mid g = s_1 \dots s_k, s_i \in S \cup S^{-1}\}$$

and

$$d_S(g, g') := \left|g^{-1}g'\right|$$

is a metric on G

- If G is a Lie group, we equip G with a left invariant Riemannian metric

$$\langle u, v \rangle_g := \left\langle L_{g^{-1}*}v, L_{g^{-1}*}w \right\rangle_{i_G}$$

 Tldr

We want to study large scale geometry of G , that is, study upto quasi-isometry.
Thus we need quasi-isometry invariants!

(Gromov: *Asymptotic invariants of infinite groups*)

growth

 Definition. Growth of fg groups

If G is finitely generated with S a finite set of generators, the **growth function** of (G, S) is

$$\begin{aligned} \rho_{G,S} : \mathbb{R} &\rightarrow \mathbb{R}_{>0} \\ r &\mapsto \left| \underbrace{\{g \in G \mid |g|_S \leq R\}}_{B(i,R)} \right| \end{aligned}$$

 Definition. Growth of Lie groups

If G is a Lie group equipped with a left-invariant Riemannian distance and h be a left Haar measure on G then the **growth function** of (G, d, h) is

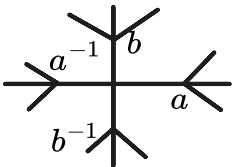
$$\begin{aligned} \rho_{G,d,h} : \mathbb{R}_{>0} &\rightarrow \mathbb{R}_{>0} \\ \rho(R) &= h(B_d(1, R)) \end{aligned}$$

- $(\mathbb{R}^n, d_{\text{std}})$ has growth

$$d(r) = cr^n$$

$$G = \pi_1 \left(\text{figure-eight} \right) \curvearrowright U \left(\text{figure-eight} \right)$$

4-valence tree



$$\text{Cal}(G, \{a, b\})$$

$$\rho(n) = 4 \sum_{k=1}^n 3^{k-1} + 1$$

Definition. Equivalence of growth functions

Let $\rho_1, \rho_2 : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$

- we say $\rho_1 \preceq \rho_2$ if $\exists a, b > 0, c, d \geq 0$

$$\rho_1(r) \leq a\rho_2(br + c) + d$$

- and $\rho_1 \simeq \rho_2$ if $\rho_1 \preceq \rho_2$ and $\rho_2 \preceq \rho_1$

Exercise

$$r^d \simeq r^D \implies d = D$$

Exercise

$$e^{\alpha r} \simeq e^{\beta r} \text{ for all } \alpha, \beta > 0$$

 Let G, H be fg or Lie groups. Then $G \text{ QI to } H$ implies $\rho_G \simeq \rho_H$.

Proposition: Let G be fg.

- if $H < G$ is a fg subgroup then $\rho_H \preceq \rho_G$
- if $N \trianglelefteq G$ then $\rho_{G/N} \preceq \rho_G$

Exercise

State and prove analogous result for Lie groups.

Proposition: Let G be a Lie group

- either
 - *polynomial growth* $\rho_G \preceq r^d$
 - *exponential growth* $\rho_G \simeq e^r$

Proposition: (Frank's lemma) Suppose G admit an automorphism $h : G \rightarrow G$ which is expanding, that is,

$$\exists \lambda > 1 : \forall g, g' \in G \ d(h(g), h(g')) > \lambda d(g, g')$$

then G has polynomial growth.

 *proof*

$$B(i, \lambda^k) \subset h^k B(i, 1)$$

- Since h is an automorphism, $h^* \nu = c \nu$
- $\implies \rho(\lambda^k) = \nu(B(i, \lambda^k)) \leq h^{k*} \nu(B(i, 1)) = c^k \nu(B(i, 1))$
- $\implies \rho(r) \preceq r^d$

Example

$$G = \left\{ \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \right\} \subset GL(n, \mathbb{R})$$

$$h_t : G \rightarrow G \\ (a_{ij}) \mapsto e^{t(j-i)}(a_{ij})$$

is an automorphism and for $t > 0$ its expanding.

Thus G has polynomial volume growth.

Proposition: If G is not unimodular, then G has exponential growth.

Example

$$G = SL(n, \mathbb{R}) = KAN$$

K is compact $\implies G$ is QI to AN which is not unimodular

Thus G has exponential growth.

 **(Wolf)** Every nilpotent (fg or Lie) group has polynomial growth.

- *proof:* by Ado-Engel's theorem

 **(Malev)** Every fg nilpotent group admits a finite index subgroup which is isomorphic to a cocompact lattice in a nilpotent Lie group.

 **(Gromov)** Let G be a fg group. If G has polynomial growth, then it admits a finite index subgroup which is nilpotent.

Exercise

$\mathbb{R} \ltimes_{\mathbb{R}^\theta} \mathbb{R}^2$ has polynomial growth (QI to $\mathbb{R}^3$) but is not nilpotent.

Proposition: A Lie group of polynomial growth is always QI to a nilpotent Lie group.

- *corollary of*

 **(Gromov)** Let G be a fg group. If G has polynomial growth, then it admits a finite index subgroup which is nilpotent.

Let G, H be fg, G is QI to H and H is nilpotent. Then G is virtually nilpotent.

Nilpotent groups and solvable groups

Examples:

- $$N := \left\{ \begin{bmatrix} 1 & & * \\ & 1 & \\ & & 1 \end{bmatrix} \right\}$$

is nilpotent

•

$$T := \begin{bmatrix} a_{11} & & * \\ & \ddots & \\ & & a_{nn} \end{bmatrix}$$

is solvable as $[T, T] = N$

 **(Gromov)** Let G be a fg group. If G has polynomial growth, then it admits a finite index subgroup which is nilpotent.

and

 **(Malev)** Every fg nilpotent group admits a finite index subgroup which is isomorphic to a cocompact lattice in a nilpotent Lie group.

implies

 **(Ado-Engel-Malev)** If a group is fg nilpotent, then it is virtually linear.

are false for fg solvable groups by following:

 **(Erscher)** G, H fg, G quasi-isometric to H but H solvable does not imply G is virtually solvable.

Consider the lamplighter group

$$G := \mathbb{Z} \ltimes_{\varphi} \bigoplus_{\mathbb{Z}} \mathbb{Z}_2$$

where

$$\begin{aligned} \varphi : \mathbb{Z} &\rightarrow \text{AutGrp} \left(\bigoplus_{\mathbb{Z}} \mathbb{Z}_2 \right) \\ (x_i) &\mapsto (x_{i+n}) \end{aligned}$$

where the multiplication is

$$(n, x)(m, y) = (n + m, x + \varphi(n)y)$$

This group is generated by

$$(1, 0), (0, (\dots 0, 0, 0, \underbrace{1}_{0\text{-th}}, 0, 0, 0 \dots))$$

This group is solvable as

$$[G, G] = \bigoplus_{\mathbb{Z}} \mathbb{Z}$$

A fg linear group is virtually torsion free but this group isn't. Thus this group is not virtually linear.

Distortion

Definition. (Gromov) Distortion

Let G be a fg or Lie group. An element $g \in G$ is

- **distorted** if

$$\frac{d(i, g^n)}{n} \xrightarrow{n \rightarrow \infty} 0$$

- **exponentially distorted** if $\exists c > 0$ such that $\forall n \in \mathbb{N}$

$$d(i, g^n) \leq C \log(1 + n)$$

- **infinitely distorted** if $\exists c > 0$ such that $\forall n \in \mathbb{N}$

$$d(i, g^n) \leq C$$

Examples:

- Consider $G = SL(n, \mathbb{R})$ and a norm $\| \cdot \|$ on $\text{Mat}(n, \mathbb{R})$ then

$$l(g) := \log \max\{\|g\|, \|g^{-1}\|\}$$

there exists $c \geq 1, D \geq 0$ such that $\forall g \in G$

$$\frac{1}{c}l(g) - D \leq d(i, g) \leq cl(g) + D$$

- g semi-simple $\neq i \implies g$ is non-distorted
- g unipotent $\implies g$ is exponentially distorted
- $g \in SO(n, \mathbb{R}) \implies g$ is infinitely distorted

 Suppose G is a Lie group. Then G admits a non-infinitely exponentially distorted element $\iff G$ is of exponential growth.

 **proof:** G is of polynomial growth $\iff X \in \mathfrak{g}$ has $\text{spec}(\text{ad}(X)) \subset i\mathbb{R}$

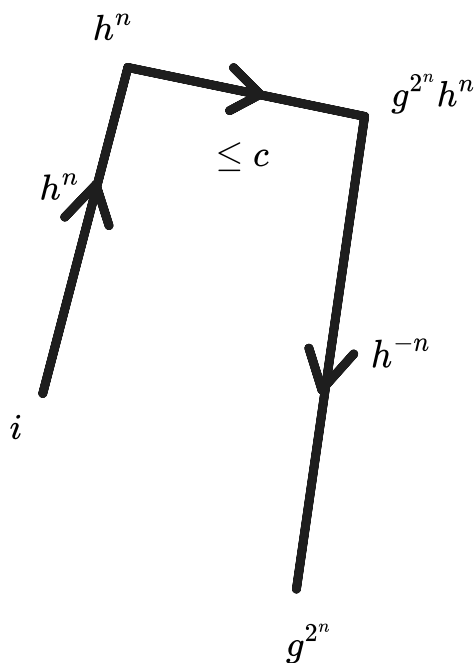
- $\implies$ every $g \in G$ is at most polynomially distorted
- if G is not of polynomial growth then $\exists H \in \mathfrak{g}$ such that $\text{ad}H$ admits an eigenvalue λ with $\Re(\lambda) \neq 0$

Proposition: Let $X \in \mathfrak{g}$ be an eigenvector of $\text{ad}H$ of eigenvalue $\lambda \in \mathbb{R} \setminus \{0\}$ then $g = \exp(X)$ is exponentially distorted.

 Assume $\lambda = \log 2$ then $h = \exp H$.

- **Proposition:** $d(i, h^{-n}g^{2^n}h^n) \leq c_0$ for $n \gg 1$

•



Exponential radical

Definition. (Osin) Exponential radical

Let G be a Lie group. Its exponential radical denoted by $R_{\exp}G$ is the set of exponentially distorted elements of G .

(Guivarc'h, Osin) Suppose G is simple connected solvable Lie group. Then $R_{\exp}G$ is a closed normal subgroup of G . Moreover it is the smallest normal closed subgroup H of G such that G/H has polynomial growth.

Here, we cannot drop the assumption of *solvable*. For example $G = SL(2, \mathbb{R})$ and $g_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $g_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ are exponential distorted but $g_1 g_2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ is not since its semi-simple.

Arghya Mondal - Mostow rigidity for closed hyperbolic 3-manifolds

Symmetric spaces of non-compact type Y are of the form

$$Y = X/\Gamma, X = G/K$$

where G is a non-compact semi-simple Lie group, no compact factor and no centre, with $K < G$ compact, $\Gamma < G$ torsion free lattice.

In its more generality, we have the rigidity theorem:

(Mostow-Prasad rigidity) Let Y_1, Y_2 are locally symmetric spaces of non-compact type, with no $\mathbb{R}H^2$ factor with an isomorphism

$$\varphi : \pi_1(Y_1) \rightarrow \pi_1(Y_2)$$

isomorphism. Then there exists an (upto scaling) isometry

$$f : Y_1 \rightarrow Y_2, f_* = \varphi$$

Mostow proved this for

- closed hyperbolic manifolds
- general case with cocompact lattices

Then Prasad proved for

- $\mathbb{Q}$ -rank 1 lattices (non-compact type in rank 1 symmetric spaces)

Let $G(\mathbb{Q})$ be an algebraic group over $\mathbb{Q}$. Then the maximal torus in $G(\mathbb{Q})$ is of dimension 1 is what " $\mathbb{Q}$ -rank 1" means.

The version we will prove is for closed hyperbolic 3-manifolds.

Let Y be a *complete* hyperbolic manifold, that is, it has constant sectional curvature -1 .

Uniformization theorem implies the universal cover $\tilde{Y}$ must be $\cong \mathbb{R}H^n$ and

$$\Gamma := \pi_1(Y, y) \curvearrowright \mathbb{R}H^n$$

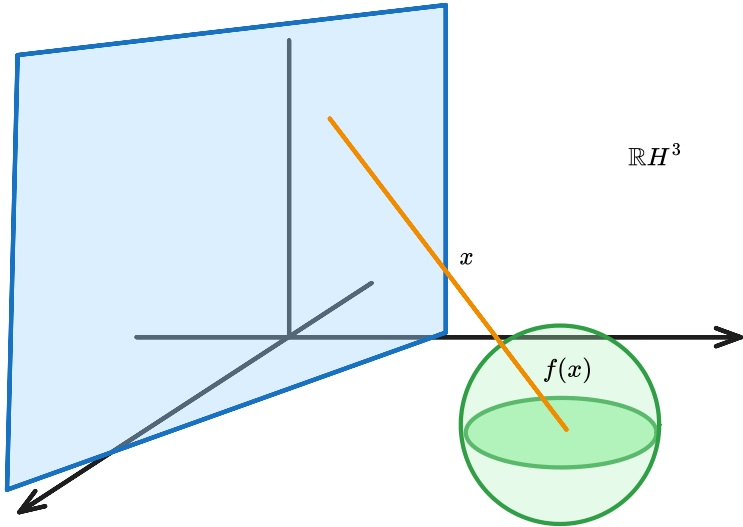
via Deck transformations this

$$Y = \frac{\mathbb{R}H^n}{\Gamma}$$

Isometries are

$$O^+(1,3) \curvearrowright \mathbb{R}H^n$$

which are generated by *reflections*



such that

$$\|f(x)\| \|x\| = R^2$$

The boundary

$$\partial \mathbb{R}H^3 := \mathbb{R}^2 \cup \{\infty\} \cong \mathbb{C}P^1$$

where Mobius transformations along with reflections act induced by the hyperbolic isometries

$$PSL(2,\mathbb{C}) \ltimes \mathbb{Z}_2 \curvearrowright \mathbb{C}P^1$$

and we have

$$\mathrm{Isom}^+(\mathbb{R}H^3) \cong SO^+(1,3) \cong PSL(2,\mathbb{C})$$

We have projection map to a totally geodesic submanifold P

$$\pi_P : H^3 \rightarrow P$$

such that the geodesic $[x, \pi_P(x)]$ is perpendicular to P for all $x \in H^3$ and

$$d(\pi_P(x), \pi_P(y)) \leq d(x, y)$$

So in particular

- $\Gamma < O^+(1, 3)$
-

☰ (Mostow rigidity for closed 3-manifold) Let M_1, M_2 be oriented closed hyperbolic 3-manifolds with $\Gamma_1 := \pi_1(M_1), \Gamma_2 := \pi_1(M_2)$ and a isomorphism

$$\varphi : \Gamma_1 \rightarrow \Gamma_2$$

Then there exists an isometry $f : M_1 \rightarrow M_2$ with $f_* = \varphi$.

📖 Tldr

1. Let

$$\varphi : \Gamma_1 \rightarrow \Gamma_2$$

be a group homomorphism. This will induce a quasi-conformal

$$\partial H^3 \rightarrow \partial H^3$$

equivariant with the action of Γ_i on the boundary.

2. Any *equivariant* quasi-conformal is induced by isometries.

We recall

📌 Definition. quasi-isometric embedding

Let X, Y are metric spaces and $k \geq 0$. Then

$$f : X \rightarrow Y$$

is called a **k -quasi isometric embedding** if

$$-k + \frac{1}{k}d(x, x') \leq d(f(x), f(x')) \leq k + d(x, x')$$

for all $x, x' \in X$.

☰ (Milnor-Schwarz) Let X be a length metric space and $\Gamma \curvearrowright X$ acts by isometries properly such that X/Γ is compact then Γ is finitely generated and

$$\begin{aligned} \Gamma &\rightarrow X \\ g &\mapsto gx \end{aligned}$$

is a quasi-isometry

This means

$$\Gamma_i \cong_{\text{quasi}} \mathbb{R}H^3$$
$$\gamma \mapsto \gamma x$$

so this induces

```
\documentclass{standalone}
\usepackage{tikz-cd}
\usepackage{amsfonts}\usepackage{amsmath}\usepackage{quiver}\usepackage{mathrsfs}

\begin{document}
\tikzcdset{scale cd/.style={every label/.append style={scale=#1},cells={nodes={scale=#1}}}}
% \begin{tikzcd}[scale cd=0.8]

%
https://q.uiver.app/#q=WzAsNCxbMCwwLCJcXEdhbW1hXzEiXSxbMSwwLCJcXEdhbW1hIl0sWzAsMSwiSF4zIl0sWzEsMSwiSF4zIl0sWzAsMSwiXFx2YXJwaGkiXSxbMCwyLCJxXzEiLDJdLFsxLDMsInFfMiJdLFsyLDMsIiIsMix7InN0eWxlIjp7ImJvZlZlZmFtZSI6ImRhczhlZCJ9fX1dXQ==
\begin{tikzcd}
{\Gamma_1} & \& {\Gamma_2} & \backslash \\
{H^3} & \& {H^3} & \\
\arrow["\varphi", from=1-1, to=1-2] & & & \\
\arrow["{q_1}", from=1-1, to=2-1] & & & \\
\arrow["{q_2}", from=1-2, to=2-2] & & & \\
\arrow[dashed, from=2-1, to=2-2] & & & \\
\end{tikzcd}

\end{document}
```

a quasi isometry

$$f : H^3 \rightarrow H^3$$

📌 Definition. quasi geodesics

A curve

$$\gamma : [a, b] \rightarrow H^3$$

is called a **(k, ϵ) -quasi geodesic** if $\forall s \in [a, b]$

$$\frac{1}{k} |s - t| - \epsilon \leq d(\gamma(s), \gamma(t)) \leq k |s - t| + \epsilon$$

and put $b = \infty$ to get a geodesic ray.

The (k, ϵ) -quasi-isometries take geodesics to (k, ϵ) -quasi geodesics.

☰ (Morse lemma) There exists a $R(k, \epsilon) > 0$ such that for every k, ϵ -quasi geodesic γ the geodesic A_γ with endpoints same as γ satisfies

$$\gamma \subset N_R(A_\gamma)$$

and

$$A_\gamma \subset N_R(\gamma)$$

Definition. Inducing map to the boundary from a quasi-isometry

Fix $x_0 \in H^3$

$$\partial f : \partial H^3 \rightarrow \partial H^3$$

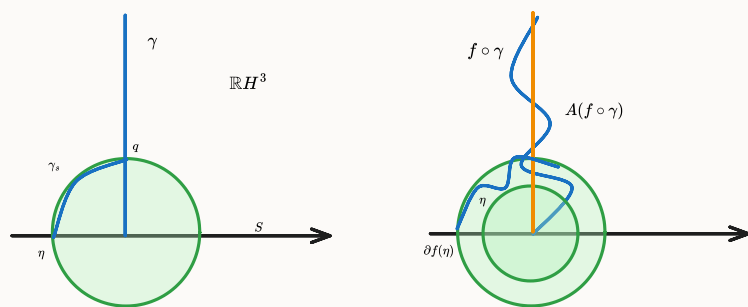
such that $s \in H^3$ there exists unique geodesic ray starting from x_0 to s then consider the geodesic $A(f \circ \gamma)$ and define

$$\partial f(s) := \lim_{t \rightarrow \infty} A(f \circ \gamma)(t)$$

Proposition: ∂f is a bijection.

💡 As f is invertible, we have a quasi-isometry f^{-1} which induces an inverse of ∂f^{-1}

Proposition: Let $f : H^3 \rightarrow H^3$ be a (k, ϵ) -quasi-isometry. There exists $B(k, \epsilon) > 0$



such that for all geodesic rays γ and any hyperplane S perpendicular to γ

$$\text{diam}(\pi_{A(f \circ \gamma)}(f(S))) < S$$

💡 Let γ_s be the geodesic ray in S from q to s with endpoint η

• ...

Proposition: ∂f is continuous for a quasi-isometry $f : H^3 \rightarrow H^3$

1. <https://people.math.osu.edu/george.924/Ehresmann%20Theorem> ↩

2. A counterexample [Rigidity of compact complex manifolds](#) ↩

Midwest 25 Summer School: Karen Butt - Entropies of negatively curved surfaces

#talk

✧ Abstract

Speaker: Karen Butt

Title: Entropies of negatively curved surfaces

Abstract: I will discuss two notions of entropy for the geodesic flow: the topological entropy and the measure-theoretic entropy with respect to the Liouville measure. How these dynamical invariants relate to the underlying Riemannian metric has long been of interest. For negatively curved surfaces, Katok proved that equality of the Liouville and topological entropies characterizes metrics of constant negative curvature. In this setting, Manning asked how these two entropies vary along the normalized Ricci flow; more specifically, he proved the topological entropy is monotonic. The main result of this talk, joint with Erchenko, Humbert, and Mitsutani, is that the same is true for the Liouville entropy. In addition to geometric and dynamical methods, we use microlocal analysis to differentiate the Liouville entropy with respect to a conformal perturbation of the metric.



Karen Butt

#people

[2504.07290] [Monotonicity of the Liouville entropy along the Ricci flow on surfaces](#)

💬 Quote

Hi Rupadarshi,

Yes, you can check out [this book \(still in progress\)](#) by Amie Wilkinson.

See, section 2.3 in the introduction for a discussion of entropy rigidity, and there's a whole chapter on it later on. I haven't read the book carefully, but I don't think you need to read the entire book if you are interested in the entropy rigidity specifically. Maybe start in that chapter and work backwards as needed.

Karen

Fanny Kassel - Geometric structures and representations of discrete groups

[#paper](#)

[#talk](#)

- ihes.fr/~kassel/Rio.pdf,
- [Geometric structures and representations of discrete groups – Fanny Kassel – ICM2018](#)

Sandip Singh - Arithmeticity and Thinness of Hypergeometric Groups

#talk

#people Sandip Singh

[Hypergeometric Groups and Their Arithmeticity - Sandip Singh, IIT Bombay](#)

✧ Abstract

[1]

Dr. Sandip Singh
IIT Bombay

3*1.5h=4.5h

Title: Hypergeometric Groups and their Arithmeticity and Thinness

Topics: A hypergeometric group $\Gamma(f, g)$ is a subgroup of $GL_n(\mathbb{C})$ generated by the companion matrices of two monic coprime polynomials f and g of degree n . It arises as the monodromy group of a hypergeometric differential equation, and if the defining polynomials f and g are self-reciprocal and form a primitive pair, then the Zariski closure G of $\Gamma(f, g)$ inside $GL_n(\mathbb{C})$ is either the symplectic group Sp_n or the orthogonal group O_n . It is immediate that when the defining polynomials f, g also have integer coefficients (in particular, when they are products of the cyclotomic polynomials), the associated hypergeometric group $\Gamma(f, g) \subseteq G(\mathbb{Z})$. In this case, the hypergeometric group $\Gamma(f, g)$ is called arithmetic if it has finite index inside $G(\mathbb{Z})$, and thin, otherwise. The question of determining the arithmeticity and thinness of the hypergeometric groups has been of great interest since the beginning of the last decade. In this mini-course, we will provide all the details about the hypergeometric groups and discuss the progress in answering the arithmeticity and thinness question for hypergeometric groups.

1. [National Centre for Mathematics | A joint centre of IIT Bombay and TIFR, Mumbai](#) ↩

Amie Wilkinson - Geometry, Dynamics, and Rigidity

[#book/draft](#)

[DGRBookCurrent.pdf](#), [DGRBookCurrent.pdf](#)

references

Karen Butt - Entropies of negatively curved surfaces

[Midwest 25 Summer School](#): Karen Butt - Entropies of negatively curved surfaces

[#talk](#)

Abstract

Speaker: Karen Butt

Title: Entropies of negatively curved surfaces

Abstract: I will discuss two notions of entropy for the geodesic flow: the topological entropy and the measure-theoretic entropy with respect to the Liouville measure. How these dynamical invariants relate to the underlying Riemannian metric has long been of interest. For negatively curved surfaces, Katok proved that equality of the Liouville and topological entropies characterizes metrics of constant negative curvature. In this setting, Manning asked how these two entropies vary along the normalized Ricci flow; more specifically, he proved the topological entropy is monotonic. The main result of this talk, joint with Erchenko, Humbert, and Mitsutani, is that the same is true for the Liouville entropy. In addition to geometric and dynamical methods, we use microlocal analysis to differentiate the Liouville entropy with respect to a conformal perturbation of the metric.

 [Karen Butt](#) [#people](#)

[2504.07290] [Monotonicity of the Liouville entropy along the Ricci flow on surfaces](#)

Quote

Hi Rupadarshi,

Yes, you can check out [this book \(still in progress\)](#) by [Amie Wilkinson](#).

See, section 2.3 in the introduction for a discussion of entropy rigidity, and there's a whole chapter on it later on. I haven't read the book carefully, but I don't think you need to read the entire book if you are interested in the entropy rigidity specifically. Maybe start in that chapter and work backwards as needed.

Karen

Introduction

The goal of this text is to illustrate how tools from dynamical systems can be brought to bear on rigidity questions in Riemannian geometry, in particular questions about:

- the lengths of closed geodesics in negatively curved surfaces,
- metrics on the torus without conjugate points, and
- numerical invariants of geodesic flows for negatively curved manifolds.

We will present three results that give insight into these topics: The Otal–Croke proof of Marked length spectrum rigidity for surfaces [247, 83], the Burago–Ivanov proof of the Hopf conjecture on tori [64], and A. Katok’s proof of the entropy rigidity conjecture on surfaces [184]. We state these results later in this chapter, in Section 2.

First we put these results in a broader context, with an (admittedly personal) view of the role of rigidity in dynamics and geometry. Most of the technical material presented in the next section will be revisited more formally in later chapters, and we have referenced this material accordingly. We have chosen to focus much of this discussion through the lens of surfaces, as many of the basic dynamical and geometric themes emerge already in this restricted setting.

Chapter 11

Appendix to Part 3: the role of microlocal analysis in local geometric rigidity

Questions about geodesics and their behavior in manifolds have long been approached using tools from analysis, particularly microlocal analysis. Dating back to classical studies on Zoll surfaces—compact surfaces whose geodesics are all closed and of equal length—analytical techniques by Israel Gelfand, Alan Weinstein, Guillemin and Kazhdan (e.g. [313, 153, 149] laid important foundations. Subsequent influential work of Christopher Croke, Vladimir Sharafutdinov, Gabriel Paternain, Marco Sasso, Gunther Uhlmann and others (e.g. [89, 250] has significantly advanced our understanding of geometric inverse problems, building upon classical transformations introduced by Funk and Radon (see also the discussion at the beginning of Section 3).

At the same time, methods from microlocal analysis have found remarkable applications in both classical and quantum dynamics. Nalini Anantharaman's work [9] in quantum dynamics, complemented by related research by Carlangelo Liverani and Viviane Baladi [25] on anisotropic Sobolev spaces, has demonstrated the power of microlocal analysis in establishing dynamical properties, both quantum and classical, such as quantum unique ergodicity and exponential decay of correlations for hyperbolic systems. Further extensions of these methods by Frédéric Faure and Johannes Sjöstrand [113], originating in the study of quantum chaos, have brought these tools to the state of the art.

Dave Witte Morris - Introduction to Arithmetic Groups

#book

[Dave Witte Morris - Introduction to Arithmetic Groups \(2015\).pdf](#)

✓ <https://www.youtube.com/@TheMathnetkorea/search?query=Dave%20Witte%20Morris%20>



GSG on orderability in 3-manifold groups

✧ Abstract

Speaker: [Debattam Das](#)

Title: Orderability in 3-manifolds groups

Abstract: A group is said to be left orderable if there is a left invariant order of the elements, presents in the group. In this talk, We will discuss the orderability on 3 -manifold groups and also the relation between orderability and generalised n -torsion elements. In the ending note, we will see some recent results on that.

Date: 21 Sep 2024 (Saturday)

Time: 3 30 PM - 5 00 PM

Venue: AB2 2B

We look forward to your presence. Feel free to ask questions to the speaker between and after the talk.

A left order on a group, bi order on a group [Linearly ordered group - Wikipedia](#)

$$t \in G \implies g < h \implies tg < th$$

☰ **(Conrad, 1959)** A group G is left-orderable if and only if it acts effectively on a linearly ordered set X by order-preserving bijections.

#paper [math/0211110v2] [Orderable 3-manifold groups personal.math.ubc.ca/~rolfsen/papers/ord/ord.final.pdf](https://personal.math.ubc.ca/~rolfsen/papers/ord/ord.final.pdf)
- [Steven Boyer](#), [Dale Rolfsen](#), [Bert Wiest](#)



$$G = G_1 * G_2 * \cdots * G_n$$

G is left orderable $\iff G_i$ are left orderable

(Prime decomposition theorem) M be cpt, conn, orientable 3-man then

$$M \cong M_1 \# M_2 \dots \# M_n$$

M_i prime manifold...

- prime manifold
- irreducible manifold
- reducible prime manifolds are $S^2 \times S^1$, ...

...and?

$$\pi_1(M) = \pi_1(M_1) * \pi_1(M_2) * \cdots * \pi_1(M_n)$$

- [Prime manifold - Wikipedia](#)

- For a prime manifold M possibly with bd with $H^1(M)$ is infinite then $\pi_1(M)$ is L.O.

- [Seifert fiber space - Wikipedia](#)

- classify
- $\pi_1(M)$
- horizontal foliation on SFS is 2-foliation which is transverse to a ?

- M be cpt, connected SFS, orientable then M admits transverse foliations $\implies \pi_1(M)$ is L.O.

- generalized torsion elements and Teragaento conjecure
- fund grp of hyperbolic 3-manifolds

$$\pi_1(M^3_{p,q,m}) = \langle t, a, b \mid t^{-1}at = aba^{m-1}, t^{-1}bt = a^{-1}, t^{-p} = (aba^{-1}b^{-1})^q \rangle$$

it not L.O. by thw conjecture? where the torsion element is

$$t^{p^2}$$

- the gen 2-tosion element in (M, ϕ)
- [Hyperbolic group - Wikipedia](#)
- ams.org/journals/jams/2003-16-03/S0894-0347-03-00426-0/S0894-0347-03-00426-0.pdf
- [Hyperbolic group - Wikipedia](#)

Boyer (2005) - Orderable 3-manifold groups

#paper

[math/0211110v2] Orderable 3-manifold groups personal.math.ubc.ca/~rolfsen/papers/ord/ord.final.pdf - Steven Boyer, Dale Rolfsen, Bert Wiest

Torsion elements in 3-manifold groups (2024)

#paper

[2406.03754] Classification of generalized torsion elements of order two in 3-manifold groups - Keisuke Himeno, Kimihiko Motegi, Masakazu Teragaito (2024)

#paper

[2403.01541v3] Reversible and other generalised torsion elements in Seifert-fibered spaces - Anushree Das, Debattam Das (2024)

GSG on liftable mapping class groups

Quote

Dear all,

Everyone is invited to the next talk of the Graduate Students' Group seminar, whose details are as follows:

Speaker: Pankaj Kapdi

Title:- Generating the liftable mapping class groups of regular $\mathbb{Z}n$ -covers of the closed and oriented surface S_g of genus $g > 0$.

Abstract: Consider a regular cover $p:S \rightarrow S_g$ of the closed and oriented surface S_g of genus $g > 0$ with the Deck group $\mathbb{Z}n$. The subgroup of the mapping class group $Mod(S_g)$ consisting of mapping classes represented by homeomorphisms that lift under the cover p is called the liftable mapping class group, denoted by $LModp(S_g)$, associated with the cover p . It is known that $LModp(S_g)$ is finitely generated. The question is, can one derive a finite generating set of $LModp(S_g)$. Consider the symplectic representation $\psi:Mod(S_g) \rightarrow Sp(2g;\mathbb{Z})$ of $Mod(S_g)$. The kernel of ψ is known as the Torelli group denoted by $\mathcal{J}(S_g)$. For $g \neq 2$, it is known that $\mathcal{J}(S_g)$ is finitely generated. Hence, for $g \neq 2$, one can obtain a finite generating set for $LModp(S_g)$ by combining a finite generating set of $\mathcal{J}(S_g)$ and lifting a finite generating set of $\psi(LModp(S_g))$. *This approach does not work for $g=2$ as $\mathcal{J}(S_g)$ is not finitely generated.* In this talk, I will discuss a method to obtain a finite generating set for $LModp(S_2)$.

Date: 11 Jan 2025 (Saturday)

Time: 11 00 AM - 12 00 PM

Venue: AB2 2A/2B

We look forward to your presence. Feel free to ask questions to the speaker between and after the talk.

Regards,

Ananya Gaur

Member, Organizing Committee

Graduate Students Group

PS: Please visit <https://sites.google.com/view/iiserm-gsg/home> for the details of the past talks. For queries and suggestions, you may contact :

Debattam Das- ph20010@iisermohali.ac.in

Shiva Barman-mp21014@iisermohali.ac.in

Ayush khare- mp20004@iisermohali.ac.in

Ananya Gaur- mp21004@iisermohali.ac.in

Let

$$p : \tilde{X} \rightarrow X$$

be a **regular** covering.

Lift of a homeomorphism

$$f : X \rightarrow X$$

is such that

$$\begin{array}{c} \tilde{X} \xrightarrow{f} \tilde{X} \\ \downarrow p \\ X \end{array}$$

Lifting criteria:

$$f:Y\rightarrow X$$

lifts to

$$\tilde{f}:Y\rightarrow \tilde{X}$$

$$\Longleftrightarrow f_*\pi_1(Y)\leq p_*\pi_1(\tilde{X})$$

If $\text{Deck}(p)$ is Abelian then $\pi_1(X)\rightarrow \text{Deck}(p)$ factors through H_1

$$\frac{\text{Homeo}^+(S_g)}{\text{isotopy}} =: \text{Mod}(S_g)$$

and the liftable mapping classes

$$\text{LMod}_p(S_g)$$

 $\text{Mod}(S_g)$ is finitely presented

$$\text{Mod}(S_g)=\langle \text{Dehn twists along } a_i,b_i,c_j \mid 1\leq i\leq g, 1\leq j\leq g-1\rangle$$

Lot of the Dehn twists *lift*.

For $g=2$ we may **conjecture**

$$\text{LMod}(S_2)=\langle T_a,T_b^3,T_c,T_d,T_e\rangle$$

symplectic rep of Mod

$$\text{Mod}(S_g)\rightarrow \text{Aut}(H_1(S_g,\mathbb{Z}))$$

where the automorphisms are $2g\times 2g$ integer matrices, but they actually go inside the **symplectic group**

$$\text{Mod}(S_g)\rightarrow Sp(2g,\mathbb{Z})$$

its kernel is called the **Torelli group** $I(S_g)$

$$I(S_g)\leq \text{LMod}(S_g)$$

lifting criteria

Symplectic criteria:

$$f_{\#}=(a_{ij})$$

lifts $\Longleftrightarrow n\mid a_{2j}$ for $j\neq 2$ and $\gcd(a_{22},n)=1$

By symplectic crietria,

$$\mathrm{Mod}(S_g)[n] \leq \mathrm{LMod}_p(S_g)$$

index

$$[\mathrm{Mod}(S_g) : \mathrm{LMod}(S_g)] = [\mathrm{Sp}(2g, \mathbb{Z}_n) : \psi_n] < \infty$$

generating sets

- can't use symplectic rep?

- $$1 \rightarrow I(S_g) \rightarrow \mathrm{LMod}(S_g) \rightarrow \psi(\mathrm{LMod}_p(S_g)) \rightarrow 1$$

We know some generating sets of $I(S_g)$

motivation

1971, Birman-Hilden

Young Mathematicians' Symposium 2025, IISER Mohali

- 09/05/2025 - 11/05/2025
- Venue: 5A, 5B (5th floor) Academic Block II

The Young Mathematicians' Symposium is an annual event that celebrates the vibrant research culture within our department. It serves as a platform for our PhD scholars and postdoctoral fellows to showcase their work, engage with peers, and spark new ideas through collaboration.

Like every year, the symposium will feature a series of plenary research talks by distinguished invited speakers. Alongside these, we proudly present contributed talks by our own students and postdoctoral fellows, highlighting the diverse and cutting-edge work happening within our community.

[YoMathS' 25](#)

Young Mathematicians' Symposium 2025
Department of Mathematical Sciences
IISER Mohali

Schedule: Day I (9th May 2025)

Day I: 9th May 2025

9:30-9:45 Registration and Inauguration

9:45-10:15 Pabitra Barman: Dominating surface-group representations in $\mathrm{PSL}_n(\mathbb{C})$ and $\mathrm{PU}(2, 1)$.

Abstract. Let S be a connected, oriented, punctured surface of negative Euler characteristic. In this talk, we present two comparison results for representations of $\pi_1(S)$ into higher-rank Lie groups. First, we show that a generic representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{C})$ is dominated by a ‘positive’ representation $\rho_0 : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ in both the Hilbert length spectrum and the translation length spectrum in the symmetric space $\mathbb{X}_n = \mathrm{PSL}_n(\mathbb{C})/\mathrm{PSU}(n)$, while preserving the lengths of the peripheral curves. This is a joint work with Subhojoy Gupta. Second, we extend this perspective to complex hyperbolic geometry: we show that a T -bent representation $\rho : \pi_1(S) \rightarrow \mathrm{PU}(2, 1)$ is dominated by a discrete and faithful representation $\rho_0 : \pi_1(S) \rightarrow \mathrm{PO}(2, 1)$ in the Bergman translation length spectrum, again preserving the lengths of peripheral curves. This is a joint work with Krishnendu Gongopadhyay. These results offer new insights into the role of positivity in higher-rank Teichmüller theory and complex hyperbolic geometry.

10:15-10:45 Dr. Neeraj Kumar Dhanwani: Generating the Lifiable Mapping Class Groups.

Abstract. For a closed orientable surface S_g , the mapping class group $\mathrm{Mod}(S_g)$ is defined as isotopy classes of orientation-preserving homeomorphisms of S_g . A mapping class is called liftable if it has a representative homeomorphism that lifts under a regular cover. In this talk, we discuss a generating set for a liftable mapping class group under k -sheeted regular free covers $p : S_{k(g-1)+1} \rightarrow S_g$.

10:45-11:00

Tea/Coffee Break

9 May 2025	Event/speaker	Title and abstract
9:00-9:30	Inauguration + registration	

9 May 2025	Event/speaker	Title and abstract
9:30-10:30	Prof. Mahuya Datta, ISI Kolkata	Title: Foliations with geometric structures Abstract: A foliation F on a manifold M gives a decomposition of M into immersed submanifolds L, called leaves. The simplest example of k-dimensional foliation on $\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^{n-k}$ is given by the leaves $\mathbb{R} \times \{y\}$, y in $\mathbb{R}^{n-k}$. If M is of dimension n, then a foliation on M is locally modelled on one of the above type of foliations on $\mathbb{R}^n$. In the first half of the talk, we will discuss the main questions and results in foliation theory after briefly introducing the basic concepts. In the second half of the talk we will discuss foliations with additional geometric structures on the leaves, focussing mainly on symplectic foliations.
10:30-11:00	Tea/Coffee break	

9 May 2025	Event/speaker	Title and abstract
11:00-11:30	Pabitra Barman	Title: Dominating Surface-Group Representations in $\mathrm{PSL}_n(\mathbb{C})$ and $\mathrm{PU}(2, 1)$. Abstract: Let S be a connected, oriented, punctured surface of negative Euler characteristic. In this talk, we present two comparison results for representations of $\pi_1(S)$ into higher-rank Lie groups. First, we show that a generic representation $\rho : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{C})$ is dominated by a ‘positive’ representation $\rho_0 : \pi_1(S) \rightarrow \mathrm{PSL}_n(\mathbb{R})$ in both the Hilbert length spectrum and the translation length spectrum in the symmetric space $X_n = \mathrm{PSL}_n(\mathbb{C})/\mathrm{PSU}(n)$, while preserving the lengths of the peripheral curves. This is a joint work with Subhojoy Gupta. Second, we extend this perspective to complex hyperbolic geometry: we show that a T-bent representation $\rho : \pi_1(S) \rightarrow \mathrm{PU}(2, 1)$ is dominated by a discrete and faithful representation $\rho_0 : \pi_1(S) \rightarrow \mathrm{PO}(2, 1)$ in the Bergman translation length spectrum, again preserving the lengths of peripheral curves. This is a joint work with Krishnendu Gongopadhyay. These results offer new insights into the role of positivity in higher-rank Teichmüller theory and complex hyperbolic geometry.
11:30-12:00	Lokenath Kundu	Title: The Dehn Function of Palindromic Automorphism Group of Free Group Abstract: The palindromic automorphism group $\Pi A(F_n)$ of a free group of rank n is a subgroup of $\mathrm{Aut}(F_n)$. The Dehn function of the group $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ is exponential for $n \geq 3$ and is known from the work of Bridson and Vogtmann. In this talk, we present a joint work with Krishnendu Gongopadhyay in which we determine the Dehn function of the palindromic automorphism group of $\mathrm{Aut}(F_n)$ for $n \geq 3$.

9 May 2025	Event/speaker	Title and abstract
12:00-12:30	Rahul Mondal	Title: Kulkarni Limit Sets for Cyclic Quaternionic Projective Groups Abstract: In this talk, I will present results on the dynamics of cyclic subgroups of $\mathrm{PSL}(n + 1, \mathbb{H})$ acting on the quaternionic projective space $\mathbb{P}^n_{\mathbb{H}}$, with particular focus on their Kulkarni limit sets. The elements of such groups have been classified as elliptic, loxodromic, or parabolic based on their spectral properties and Jordan forms. Building on this classification, we describe the structure of the corresponding Kulkarni limit sets and the associated domains of discontinuity. This is a joint work with Krishnendu Gongopadhyay and Sandipan Dutta.
12:30-2:00	Lunch break	
2:00-3:00	Prof. Anand Sawant, ITFR Mumbai	Title: R-equivalence in algebraic groups from the perspective of A^1-homotopy theory. Abstract: The group of R-equivalence classes in an algebraic group is an important classical invariant in the study of their near-rationality properties. I will describe its relationship with the sheaves of naive and genuine A^1-connected components and discuss how some of the classical questions involving R-equivalence can be approached using modern techniques. The talk will be based on some old, recent and ongoing joint works with Chetan Balwe and Amit Hogadi.

9 May 2025	Event/speaker	Title and abstract
3:00-3:30	Shruti Rastogi	Title: Motivic Intersection Complex of a Threefold Abstract: The triangulated category of motivic sheaves $DM(X)$ over a scheme X, with rational coefficients, admits a realization functor to the bounded derived category of mixed constructible sheaves $D^b_m(X, \mathbb{Q}_l)$. The latter is equipped with a perverse and a weight structure, as well as the S. Morel's t-structure. In this talk, we explore an analogue of Morel's t-structure within a suitable subcategory of $DM(X)$. A key object in the theory of perverse sheaves is the intersection complex IC_X, which appears as a simple object in the heart of the perverse t-structure and can also be characterized using Morel's weight truncation functors. Motivated by this perspective, we define its motivic analogue EM_X for a three-dimensional variety X, using the motivic version of Morel's truncation. We show that EM_X defines a pure motive, and more precisely a Chow motive offering a parallel to the classical intersection complex in the motivic setting.
3:30-4:00	Tea/Coffee break	

9 May 2025	Event/speaker	Title and abstract
4:00-4:30	Nidhi Gupta	Title: A^1-Connected Components of Affine Quadrics Abstract: Let X be a variety over a field k. Denote by $\pi^{A^1}_0(X)$ the sheaf of A^1-connected components of X, and by $S(X)$ the sheaf of naive A^1-connected components of X. A result of Balwe, Rani, and Sawant shows that for any field extension F/k, the canonical map $\pi^{A^1}_0(X)(F) \rightarrow \lim_n S^n(X)(F)$ is an isomorphism. In this talk, we show that for any field F/k and any smooth affine quadratic hypersurface X, this isomorphism stabilizes at $n=2$. That is, $\pi^{A^1}_0(X)(F) = S^2(X)(F).$ As an application, we combine this result with Morel's characterization of A^1-connected spaces in terms of the triviality of field-valued sections of $\pi^{A^1}_0$ to give a complete characterization of A^1-connected affine quadrics. This talk is based on joint work with Dr. Chetan Balwe.

9 May 2025	Event/speaker	Title and abstract
4:30-5:00	Sohan Ghosh	Title: Euler Characteristics of weight 1 modular forms Abstract: Let E be an elliptic curve over $\mathbb{Q}$. The Selmer group is a fundamental cohomological tool capturing deep arithmetic information about elliptic curves. Fix an odd prime p, and let $\mathbb{Z}_p$ denote the ring of p-adic integers. For each $n \geq 1$, let ζ_{p^n} be a primitive p^n-th root of unity in $\overline{\mathbb{Q}}$. The cyclotomic $\mathbb{Z}_p$-extension, denoted by $\mathbb{Q}_{\text{cyc}}$, is the unique extension of $\mathbb{Q}$ in $\mathbb{Q}(\cup_{n \geq 1} \mathbb{Z}_{p^n})$ whose Galois group $\Gamma := \text{Gal}(\mathbb{Q}_{\text{cyc}}/\mathbb{Q}) \cong \mathbb{Z}_p$. When E has good ordinary reduction at p, the Euler characteristic of the p-Selmer group over $\mathbb{Q}_{\text{cyc}}$ is conjecturally related to the special value $L_E(1)/\Omega_E$, under the Birch and Swinnerton-Dyer conjecture and some suitable assumptions. Here, $L_E(s)$ denotes the Hasse–Weil (complex) L-function of E over $\mathbb{Q}$, and Ω_E is the smallest positive real period of E. In this talk, we raise questions about how similar results might extend to modular forms of weight 1.

9 May 2025	Event/speaker	Title and abstract
5:00-5:30	Chitrarekha Sahu	Title: Picard-Vessiot Extensions with Unipotent Differential Galois Groups Abstract: A differential field is a field F equipped with a derivation $D : F \rightarrow F$, an additive map satisfying the Leibniz rule: $D(ab) = D(a)b + aD(b)$ for all $a, b \in F$. Analogous to the classical Galois theory for polynomial equations, differential Galois theory addresses a similar theory for linear differential equations. Informally, a Picard-Vessiot extension E of a differential field F is a differential field extension of F generated by the solutions of a linear homogeneous differential equation with coefficients in F. The group of all differential automorphisms of E that fix F and commute with the derivation D is called the differential Galois group of the extension. For a differential field F having an algebraically closed field of constants, we analyze the structure of Picard-Vessiot extensions of F whose differential Galois groups are unipotent linear algebraic groups, and as an application of these results we will give a generalization of Liouville's theorem [Ros68]. This talk is based on the joint work with Dr. Matthias Seiss and Dr. Varadharaj R. Srinivasan [SSS25].

10 May 2025	Event/speaker	Title and abstract
9:30-10:30	Prof. Manjunath Krishnapur, IISc	Title: On lemniscates of polynomials Abstract: The (filled) t-lemniscate of a complex polynomial is the set $\{z : \ p(z)\ \leq 1\}$. On many occasions Erdos raised fascinating questions on the extremal properties of lemniscates, most famously in an influential paper with Herzog and Piranian in 1958. Many of these questions are still unanswered. One of them is the area question: What is the minimum possible area of the 1-lemniscate of a degree n polynomial whose zeros are inside the closed unit disk? The best available upper bound was $1/(\log \log n)^{0.5+\epsilon}$ due to Wagner and a lower bound of $1/n^4$ due to Pommerenke. In joint work with Erik Lundberg and Koushik Ramachandran, we have improved these bounds to $1/\log \log n$ and $1/\log n$ respectively, but the gap still remains. The talk will survey some of the questions in the area and outline a proof of this and other related results. The lecture is meant to be accessible to anyone with a basic knowledge of real and complex analysis.
10:30-11:00	Tea/Coffee break	

10 May 2025	Event/speaker	Title and abstract
11:00-11:30	Anusha Bhattacharya	Title: Graph Discretization of Riemannian Manifolds for Spectral Approximation and Learning Applications Abstract: This talk discusses the approximation of eigenvalues and eigenfunctions of the Laplace–Beltrami operator on compact Riemannian manifolds without boundary via weighted graph discretization. Building on the work of Burago et al, I present a joint work with Dr. Soma Maity which generalizes their results to a broader class of manifolds satisfying lower bounds on the Ricci curvature and injectivity radius and upper bounds on diameter and sectional curvature. The construction yields uniform spectral convergence in this class. I will conclude by illustrating connections to spectral graph neural networks, where such discretizations underpin the use of graph Laplacian to approximate manifold-based learning architectures.
11:30-12:00	Yogesh	Title: Preferential Attachment Based on Reinforcement Abstract: We consider a preferential attachment random graph model with self-reinforcement. Each time a new vertex is added, it attaches to an existing vertex with a probability proportional to the sum of the degrees of that vertex across all previous time steps. The resulting growing graph forms a random tree whose vertex degrees increase at a polynomial rate over time. We compute the growth exponent of this model and demonstrate that it is strictly larger than the growth exponent in models without self reinforcement. Our analysis provides insights into how self-reinforcement influences network evolution and accelerates degree growth. This is joint work with Prof. Frank den Hollander.

10 May 2025	Event/speaker	Title and abstract
12:00-12:30	Partha Kumbhakar	Title: Liouville’s Theorem for Integration in Terms of Abelian Integrals and Abelian Elements. Abstract: We advance Liouville’s theorem on integration in finite terms ([1]) to include abelian integrals and abelian elements. This is a joint work with Sudip Pandit and Varadharaj Ravi Srinivasan.
12:30-2:00	Lunch break	
2:00-3:00	Prof. Tanmay Deshpande, TIFR Mumbai	Title: Character sheaves on tori over local fields Abstract: I will begin with an introduction to the theory of character sheaves and Grothendieck's sheaf-function correspondence in the simplest case, namely that of connected commutative algebraic groups (e.g. the additive group, the multiplicative group, abelian varieties) defined over finite fields and also in the case of pro-algebraic groups. I will then show that character sheaves on tori defined over local fields are naturally parametrized by inertial local Langlands parameters and will relate this result to the local Langlands correspondence for tori due to Langlands. The talk is based on joint work with Saniya Wagh.
3:00-3:15	Tea/Coffee break	
3:15-3:45	Diksha Mukhija	Title: Studying descent of Quadratic forms in characteristic 2 Abstract: Most courses on quadratic forms over fields start with the hypothesis of characteristic being different than 2. In this talk, we will work with quadratic forms over characteristic 2 fields. After reviewing some basic properties, we'll address the question of Descent. The descent problem seeks conditions under which a K-form (quadratic or bilinear) is defined over F for a field extension K/F. We'll answer it for some small dimensional cases when K is the function field of a quadric and address the difficulties we encounter while answering this question in general.

10 May 2025	Event/speaker	Title and abstract
3:45-4:15	Harish Kishnani	Title: Products of Cycle-Classes of Symmetric Groups Abstract: Abstract. Let G be a finite group and $C_1, C_2, \dots, C_k$ be non-trivial conjugacy classes of G. The product of these classes defined by $C_1C_2 \dots C_k := \{x_1x_2 \dots x_k x_i \in C_i, 1 \leq i \leq k\}$ is invariant under conjugation and thus a union of conjugacy classes of G. In particular, for an integer $k \geq 2$, the k-th power of a conjugacy class C of G, denoted by C_k, is simply the set $CC \dots C$ (k-times). The main question is, for a given collection of non-trivial conjugacy classes of a finite group G, how large is the set $C_1 \dots C_k$? In particular, when does this set cover the entire group G? It is a well-established theme in finite group theory which is very active at the current moment. The works of Bertram, Herzog, Lev, Kaplan, Guralnick, Malle, etc. are good sources of motivation for this problem. In this talk, I will first discuss some already known results in this direction. Then, a joint work with Dr. Rijubrata Kundu and Dr. Sumit C. Mishra will be discussed. In the symmetric group, we take the power of a conjugacy class of cycles of a fixed length and determine conditions under which it will cover the alternating group. We will provide a complete answer to both the conjectures of Herzog, Lev and Kaplan [Herzog, Marcel; Kaplan, Gil; Lev, Ariele, Covering the alternating groups by products of cycle classes. J. Combin. Theory Ser. A 115 (2008), no. 7, 1235–1245].

10 May 2025	Event/speaker	Title and abstract
4:15-4:45	Manujith K Michel	Title: Splitting iterative derivations on central simple algebras Abstract: The splitting of a derivation d on a k–central simple algebra A by a Picard- Vessiot extension of $(k, d k)$ is well studied in the case that the constants of $(k, d k)$ is an algebraically closed field of characteristic zero. In fact in this case the Picard-Vessiot extension K for the differential module A is the smallest no new constant extension splitting d. One can also characterize the structure of the differential algebra A using the structure of the differential Galois group of K. Similar results are obtained in positive characteristic if we use iterative derivations and Picard-Vessiot theory of ID–modules. More precisely, we show that if d is an iterative derivation on a k–central simple algebra A such that the constants of $(k, d k)$ is algebraically closed then the Picard-Vessiot extension K for the ID–module A is the smallest no new constant extension splitting d. We also characterise the structure of the ID–algebra A when the differential Galois group of K is linearly reductive or solvable.
4:45-5:30	Tea break and Poster presentation	

11 May 2025	Event/speaker	Title and abstract
9:30-10:30	Prof. Sayani Bera, IACS Kolkata	Title: Attracting Basins and Automorphisms of $\mathbb{C}^n$, $n \geq 2$ Abstract: Unlike the complex plane, the automorphisms, i.e., the bijective holomorphic maps of $\mathbb{C}^n$, $n \geq 2$ forms a complicated group (under composition). The goal of this talk will be to explore (iterative) dynamics of an automorphism of $\mathbb{C}^2$, near an attracting fixed point. Additionally, we will generalise this model for non-autonomous (random) dynamical systems of automorphisms, which leads to the solution of a long-standing open problem called Bedford's Conjecture. Also, if time permits, we will briefly emphasize on the relevant ideas that extend the above result to $\mathbb{C}^n$, $n \geq 3$. This is a joint work with Kaushal Verma.
10:30-11:00	Tea/Coffee break	
11:00-11:30	Aditya Tiwari	Title: Dirichlet Domains in Complex Hyperbolic Bidisks Abstract: The geometry of bidisks -- the product of two hyperbolic planes -- has been widely studied. In this talk, we explore a complex analogue: the product of two complex hyperbolic planes. Focusing on cyclic group actions, we investigate the structure of Dirichlet domains in this space. We describe the behaviour of equidistant surfaces, introduce the notions of visibility and invisibility in this context, and demonstrate that for loxodromic actions, the Dirichlet domain remarkably has exactly two faces. Along the way, we highlight the interplay between the product metric, holomorphic sectional curvature, and the boundary behaviour of the level sets. This exploration opens new avenues for understanding discrete group actions in higher-rank complex hyperbolic geometry. This is a joint work with Prof. Krishnendu Gongopadhyay and Dr. Lokenath Kundu.

11 May 2025	Event/speaker	Title and abstract
11:30-12:00	Tathagata Nayak	Title: On Character Variety of Anosov Representations Abstract: Let Γ be the fundamental group of a k-punctured, $k \geq 0$, closed connected orientable surface of genus $g \geq 2$. In this talk, it will be shown that the character variety of the $(Q+, Q-)$-Anosov irreducible representations, resp. the character variety of the $(P+, P-)$-Anosov Zariski dense representations of Γ into $SL(n, \mathbb{C})$, $n \geq 2$, is a complex manifold of complex dimension $(2g + k - 2)(n^2 - 1)$. For $\Gamma = \pi_1(\Sigma_g)$, these character varieties are holomorphic symplectic manifolds. This talk is based on a joint work with Prof. Krishnendu Gongopadhyay.
12:00-12:30	Rifat Siddique	Title: Character space and Gelfand type representation of locally C^*-algebra Abstract: In this talk, we present a suitable notion of character space of commutative unital locally C^*-algebra using the notion of inductive limit of topological spaces and study its properties. We prove that every commutative unital locally C^*-algebra is isomorphic (as a topological $*$-algebra) to the space of continuous functions on its character space. As a consequence, we define unital locally C^*-algebra generated by a locally bounded normal operator T and its character space is homeomorphic to the local spectrum of T. Also, the functional calculus and spectral mapping theorem are proved in this framework. The talk is based on a joint work with Dr. Santhosh Kumar Pamula.
12:30-2:00	Lunch break	

11 May 2025	Event/speaker	Title and abstract
2:00-3:00	Prof. Somnath Basu	Title: On manifolds homeomorphic to spheres. Abstract: We shall discuss Reeb's Theorem and basic differential topology of Morse functions. This was used by Milnor to prove the existence of exotic spheres in 7 dimensions. We shall propose a generalization of Reeb's Theorem and discuss a proof of it. This is joint work with Sachchidanand Prasad.
3:00-3:30	Pragya Belwal	
3:30-4:00	Tea/Coffee break	
4:00-4:30	Neeraj Kumar Dhanwani	Title: Generating the Lifiable Mapping Class Groups Abstract: For a closed orientable surface S_g, the mapping class group $\text{Mod}(S_g)$ is defined as isotopy classes of orientation-preserving homeomorphisms of S_g. A mapping class is called liftable if it has a representative homeomorphism that lifts under a regular cover. In this talk, we discuss a generating set for a liftable mapping class group under k-sheeted regular free covers $p: S_{k(g-1)+1} \rightarrow S_g$.
4:30-5:00	Deepanshi Sarraf	
5:00-5:30	Pravin Kumar	Title: Orderability of big mapping class groups. Abstract: An orderable group is a group that is totally ordered, with the ordering being invariant under left-multiplications. In this talk, we will see that a generalized ideal arc system for surfaces determines a left ordering on the mapping class group of a connected, oriented infinite-type surface with a non-empty boundary. We will also explore the conditions under which two generalized ideal arc systems induce the same left ordering. Finally, we will conclude the talk by comparing the usual topology of mapping class groups with the order topology. This is a joint work with Prof. Mahender Singh and Dr. Apeksha Sanghi.
5:30-6:00	Note of thanks and closing.	

Prof. Somnath Basu - On manifolds homeomorphic to spheres

Abstract

Abstract: We shall discuss Reeb's Theorem and basic differential topology of Morse functions. This was used by Milnor to prove the existence of exotic spheres in 7 dimensions. We shall propose a generalization of Reeb's Theorem and discuss a proof of it. This is joint work with Sachchidanand Prasad.

Quote

Dear Rupadarshi,

There is Reeb's Theorem (Milnor's book on Morse theory is the best source). There is an enhancement of this where we do not assume that the critical points are non-degenerate (Saaty's [Lectures on Modern Mathematics vol. 2](#) where Milnor has written a chapter on differential topology and it has a proof).

If we go the way of Bott, then we get to Morse-Bott theory (an article by Banyaga & Hurtubise on Morse-Bott functions, a chapter by Austin & Braam in the Floer memorial volume). Majority of Morse-Bott theory texts are written towards symplectic geometry/topology applications in mind. There is also Morse-Bott-Smale theory for infinite dimensional setting. There is also a [paper](#) by Gromoll-Meyer which says how to modify/perturb a function with isolated singularities to a Morse function. This may be adapted to Morse-Bott functions to perturb such functions to Morse functions.

Best wishes,

Somnath

Anushree Das - Volume growth of complete Riemannian manifolds under suitable curvature conditions

#talk/attended

#talk/PhD

✧ Abstract

Dear all,

You are cordially invited to attend the PHD thesis seminar by Ms. Anushree Das.

Here are the details.

Date and time: June 16 (Monday) , 2025, 4:30-5:30 PM

Venue: AB2- 5A

Title: Volume growth of complete Riemannian manifolds under suitable curvature conditions

Abstract: For an open manifold M with one end, and a function v with bounded growth of derivative, there exists a Riemannian metric of bounded geometry on M such that the volume growth function lies in the same growth class as v , as proved

by R. Grimaldi and P. Pansu. We prove this in the case of manifolds with multiple ends and call the constructed metrics Grimaldi-Pansu metrics. We give uniform bounds for the volume growth function of these metrics in terms of the given bgd-function in the case of a certain class of manifolds, and investigate some geometric properties of the Grimaldi-Pansu metrics. Next,

we let M be an open manifold of dimension at least 3, which admits a complete metric of positive scalar curvature. For a function v with bounded

growth of derivative, whether M admits a metric of positive scalar curvature

with volume growth of the same growth type as v is unknown. We answer

this question positively in the case of manifolds, which are infinite connected

sums of closed manifolds that admit metrics of positive scalar curvature. We

generalize this result to manifolds, which are infinite connected sums of similar

closed manifolds along lower-dimensional spheres. We crucially use the Gromov-Lawson construction of metrics of positive scalar curvature on a connected sum to construct the required metrics.

With best regards,

Soma Maity,

Assistant Professor,

IISER Mohali,

Mohali, Punjab 140306

- [\[2309.06868\] Volume growth on manifolds with more than one end](#)
- [\[2410.04121\] Volume growth functions of complete Riemannian manifolds with positive scalar curvature](#)

Inst Fourier - Low dim topology

Arunima Ray - Surfaces in 4-manifolds

[Arunima Ray - Surfaces in 4-manifolds 1 - YouTube](#)

3-manifolds

3-Manifolds.

For 3-manifold theory there are several books:

- W Thurston. *Three-Dimensional Geometry and Topology*. Princeton University Press, 1997. [\$55]

A geometric introduction by the master. Also useful for the geometry of surfaces.

- A Hatcher. *Basic Topology of 3-Manifolds*. Unpublished notes available online at <http://www.math.cornell.edu/~hatcher>

The more classical topological aspects of 3-manifold theory.

- J Hempel. *3-Manifolds*. Annals of Math Studies 86. Princeton University Press, 1976. [\$30]
- P Scott. *Geometries of 3-manifolds*. Bull. London Math. Soc. 15: 401-487, 1983.

A clear presentation of seven of Thurston's eight possible geometric structures on 3-manifolds, all but hyperbolic geometry, the most subtle case by far.

- N Saveliev. *Invariants for Homology 3-Spheres*. Springer, 2002. [\$99]
- M Kapovich. *Hyperbolic Manifolds and Discrete Groups*. Birkhäuser, 2001. [\$77] Knot Theory.

https://people.dm.unipi.it/martelli/Geometric_topology.pdf

<https://math.iisc.ac.in/~gadgil/expos/surgery.pdf>

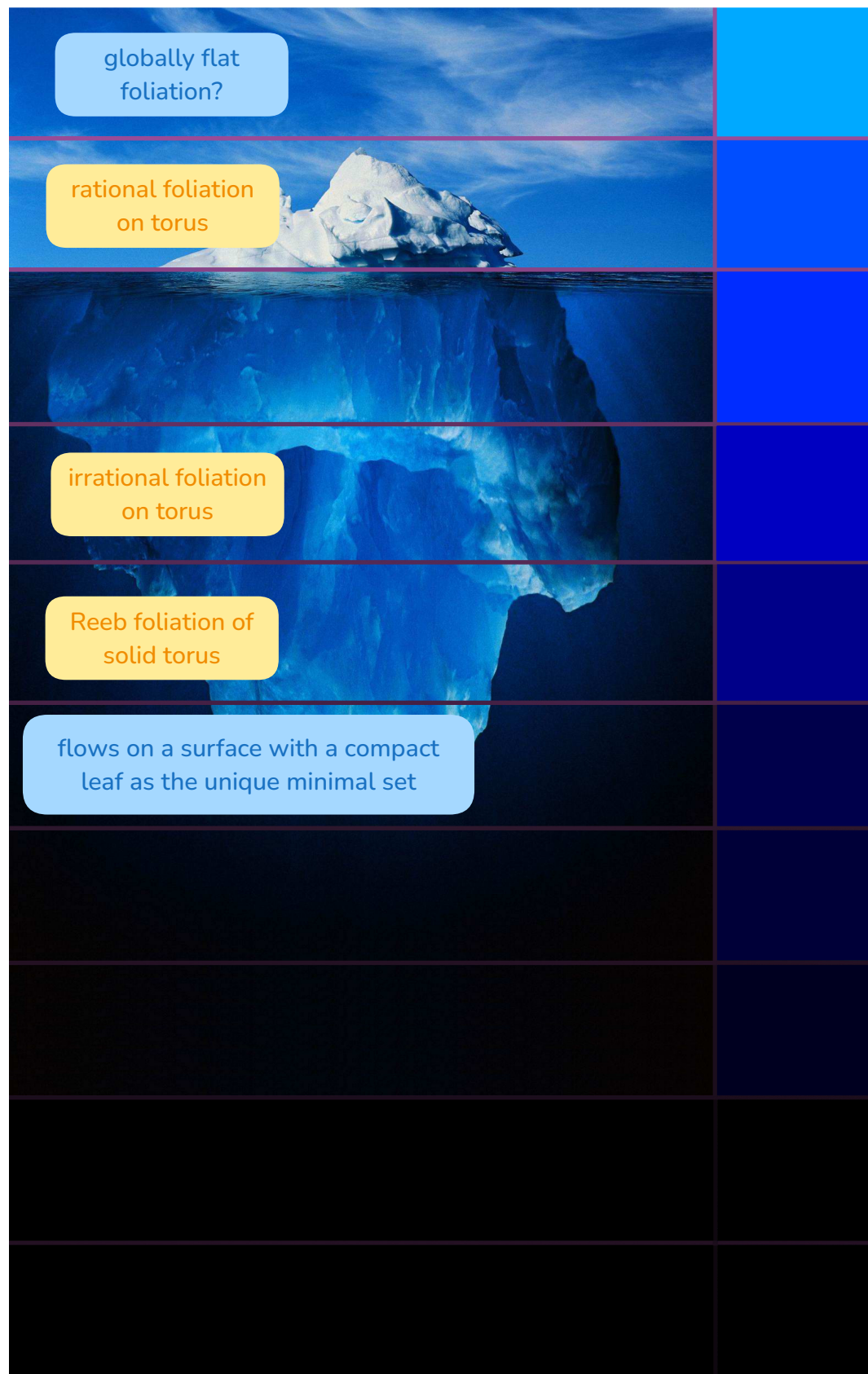
- [Dehn surgery - Wikipedia](#)
- [sett.Man.R.group action manifolds](#)
- [sett.Man.R Homog.model geometry](#)

Foliations



#reference

- <https://homepages.math.uic.edu/~hurder/papers/70manuscript.pdf>
- <https://webpace.science.uu.nl/~ban00101/foliations2006/foliations.pdf>
- <https://mathoverflow.net/questions/230426/what-is-a-foliation-and-why-should-i-care>
- https://www.math.unl.edu/~mbrittenham2/papers/folnotes/lectures_1_to_10.pdf
- <https://www.youtube.com/watch?v=epEaXLS7ajM>
- [Adolfo Guillot: Complete holomorphic vector fields and their singular points - lecture 3](#)



leaves are parallel and contractible, hence the foliation has no germinal holonomy

Every orbit limits to the circle, which is the forward (and backward) limit set for all leaves.

tangent distributions and involutivity

$$F(t, x(t), x'(t)) = 0 \iff \begin{cases} \gamma(t) = (t, x(t), p(t)) \\ \gamma \subset F^{-1}(0) \\ (dx - p dt)(\gamma') = 0 \end{cases}$$

Definition. Tangent distributions

A (topological/smooth) **tangent k -distribution on a smooth manifold M** is a rank- k (topological/smooth) subbundle of TM .

Given a smooth tangent distribution $D \subseteq TM$, a nonempty immersed submanifold $N \subseteq M$ is an **integral manifold**

of D if

$$T_p N = D_p$$

at each $p \in N$.

foliations

Definition. Foliation on a smooth manifold

Let M be a smooth n -manifold and $\mathcal{F}$ be a collection of k -submanifolds of M . A smooth chart onto a rectangle

$$\varphi : U \subset M \rightarrow \prod_{i=1}^n [a_i, b_i]$$

is a **flat chart for $\mathcal{F}$** if each submanifold in $\mathcal{F}$ intersects U either in the empty set or in a countable union of k -dimensional slices of the form

$$\{x^{k+i} = c_i \mid i \geq 1\}$$

A **k -foliation on smooth manifold M** is a collection $\mathcal{F}$ of disjoint, connected, nonempty immersed sub- k -manifolds of M (called **leaves**) whose union is M and such that in a neighborhood of each $p \in M$ there exists a flat chart for $\mathcal{F}$.

involutive tangent distributions $\leftrightarrow$ foliations

18.966 Geometry of Manifolds II

* Abstract

Differential forms, introduction to Lie groups, the DeRham theorem, Riemannian manifolds, curvature, the Hodge theory. [18.966](#) is a continuation of [18.965](#) and focuses more deeply on various aspects of the geometry of manifolds. Contents vary from year to year, and can range from Riemannian geometry (curvature, holonomy) to symplectic geometry, complex geometry and Hodge-Kähler theory, or smooth manifold topology. Prior exposure to calculus on manifolds, as in [18.952](#), recommended.

- 1992
 - [Guillemin, V., Course 18.966 – Geometry of Manifolds, M.I.T., Spring of 1992](#) from [Lectures on Symplectic Geometry](#)
- 1998
 - [T. Mrowka, 18.966 Lecture Notes, Spring 1998](#) from [Notes on J-Holomorphic Maps](#)
- 2004 (Alan Edelman)
- 2005 (Tomasz Mrowka)
- 2006 (Victor Guillemin)
- 2007 (Denis Auroux)
 - [Geometry of Manifolds](#) | [Mathematics](#) | [MIT OpenCourseWare](#)
- 2008 (Denis Auroux)
- 2009 (Tomasz Mrowka)
- 2010 (Tomasz Mrowka)
- 2011 (Peter Ozsváth)
- 2012 (Colding)
- 2013
 - [18.966 -- Spring 2013 \(Guth\)](#)
 - The theme of the class is the connection between analysis on the one hand and the topology of manifolds on the other hand. There are three main topics.
 - The first is transversality, covering Sard's theorem and its applications in topology. The applications include degrees of maps, linking numbers, and the Hopf invariant.
 - The second topic is vector bundles, connections, and characteristic classes. We will study the Euler class, and Chern and Pontryagin classes. One of the main results we will study is the Gauss-Bonnet-Chern theorem.
 - The third topic is Morse theory, connecting the critical points of a function to the topology of the manifold. We will begin with Morse theory on finite dimensional manifolds, and then study Morse theory on the space of loops on a manifold, building up to a proof of the Bott periodicity theorem on the homotopy groups of the unitary group.
 - **Texts:** In the first unit, we will use the book *Topology from the Differentiable Viewpoint*, by John Milnor. In the third unit, we will use the book *Morse Theory*, also by John Milnor. For the second unit, we will use in-class lectures, and I may post some references on the webpage.
- 2014
 - [18.966 -- Spring 2014 \(Murphy\)](#)
 - Contact geometry is a topic at the intersection of many fields, closely related to classical dynamics, complex analysis, and geometric topology. It is best thought of as the odd-dimensional cousin to

symplectic geometry. The beginning of the class will cover the basic theory, and afterwards we will specialize to the 3-dimensional case, where a number of additional tools strengthen our knowledge of that dimension (particularly Giroux flexibility).

- Contact isotopies. Gray stability, Darboux, and Legendrian neighborhood theorems. Connections between contact and symplectic geometry. Geometry of hypersurfaces.
- Tight contact structures and Bennequin's inequality. Applications of Bishop disk fillings. Giroux flexibility. Eliashberg's proof of Cerf's theorem. Connect sum decompositions of tight contact manifolds. Classification of overtwisted contact structures. Legendrian knots.
- As time allows confoliation theory and/or topics from high dimensional contact geometry will be covered.
- **Text Book:** No official text, but Geiges' book is an excellent resource that covers most of the topics discussed. There will also be class notes.

- <https://math.mit.edu/classes/18.966/2014SP/notes.pdf>

- found the link in [dg.differential geometry - Characterization of contact vector fields - MathOverflow](#)

- 2015 (Colding)

- 2016 (Paul Seidel)

- 2017 (Minicozzi)

- 2018

- [18.966 -- Spring 2018 \(Melrose\)](#)

- 2019

- [18.966 -- Spring 2019 \(Collins\)](#)

- [Math 18.966: Geometry of Manifolds II](#)

- [18.966_2019Spring_Comparison Geometry and Ricci Spaces_Collins.pdf - Google Drive](#) by [Ao Sun's homepage](#) - [Notes](#)

-  **This is the course note of 18.966 at MIT taught by Professor Tristan Collins in 2019 Spring. The note was taken together with Feng Gui. This note contains basic materials of comparison geometry and Ricci Spaces.**

- 2020 (Minicozzi)

- [18-966-full.pdf - Google Drive](#) by [Sinho Chewi](#)

- 2021 (Colding)

- <https://math.mit.edu/~tangkai/note/HE.pdf> by [Tang-Kai Lee](#)

- [【MIT数学课程】流形上的几何: 18.966 Geometry of Manifolds II 2021 Spring_哔哩哔哩_bilibili](#)

- 2022

- [18.966 -- Spring 2022 \(Colding\)](#)

- <https://math.mit.edu/classes/18.966/2022SP/>

- Form and shape can be described by differential equations. Many of these equations originate in various branches of science and engineering. They are fundamental and, in a sense, canonical. The fact that they make sense geometrically means that they are relevant everywhere and have fundamental properties that appear over and over again in many settings. Understanding them requires simultaneous insight into analysis and geometry and the interplay between these.
 - In this class we will discuss a number of different ideas and estimates that have a wide range of applications to many fields including geometry, analysis, probability and applied mathematics. Common for them all is that they originate in geometry.
 - Topics will include (but not restricted to):
 - Continues and discrete Laplacian.
 - Drift Laplacian and weighted inequalities.

- Gradient estimates.
- Harnack inequalities.
- Sharp gradient estimate and monotonicity.
- L2 eigenfunctions.
- Ornstein-Uhlenbeck operator and Hermite polynomials.
- Li-Yau differential Harnack inequality.
- Hamilton's matrix maximum principle.
- Perelman's monotonicity.
- 2023
 - [18.966 -- Spring 2023 \(Colding\)](#)
 - [Non-Blinding | MIT Admissions](#) [The Meta Home](#) [Paige](#)
- 2024
 - [18.966 -- Spring 2024 \(Minicozzi\)](#)
- Not offered in Spring 2025.

more MIT18.89x courses

- 18.969
 - [18.969 - Topics in Geometry - Spring 2009](#)
- 18.965
 - holdenlee.github.io/coursework/math/18.965/main.pdf
 - mathweb.ucsd.edu/~ebelmont/965-notes.pdf

Chow Lu Ni - Hamilton's Ricci flow

chapter 2

standard shrinking sphere

Proposition: Let $M = S^n$ and g_{S^n} be the standard metric on S^n . Then for $r_0 > 0$

$$g_0 := r_0^2 g_{S^n}$$
$$g(t) := (r_0^2 - 2(n-1)t) g_{S^n}$$

is the solution (called **standard shrinking n -sphere** of initial radius r_0) to *Ricci flow* with $g(0) = g_0$ for maximal time interval

$$(-\infty, r_0^2/2(n-1))$$

Gerhard Huisken - Ricci flow

<https://www.mfo.de/about-the-institute/staff/prof-dr-gerhard-huisken/lectures/introduction-to-ricci-flow>

Einstein metrics and Ricci solitons

[1]

📌 Definition. (M, g) is an **Einstein manifold** if

$$\text{Ric}g = \lambda g$$

where $\lambda \in \mathcal{C}^\infty(M)$

- Taking trace both sides we have

$$\text{Ric}S = \lambda \dim M \implies \lambda = \frac{\text{Ric}S}{\dim M}$$

📌 Definition. **Ricci flow**

$$\frac{\partial}{\partial t} g(t) = -2\text{Ric}g(t)$$

curvature in 2 dim

The Riemann curvature tensor

$$R_{ijkl}$$

is determined by one component

$$R_{1212}$$

so if a tensor is equal to the Riemann tensor if it has the same symmetries and

1. <https://www.youtube.com/watch?v=ZUCq0oMb-FE> ↩

Potential flows

ergodic on compact Σ_c

3.7.27 Example. On $P = \mathbf{R}^{6N} = \mathbf{R}^{3N} \times \mathbf{R}^{3N}$ consider a Lagrangian of the form

$$L(q, \dot{q}) = \sum_{j=1}^N \frac{1}{2} m_j \|\dot{q}_j\|^2 - V(q_1, \dots, q_n), \quad \dot{q}_j, q_j \in \mathbf{R}^3$$

where V is symmetric, with associated energy function

$$E(q, \dot{q}) = \sum_{j=1}^N \frac{1}{2} m_j \|\dot{q}_j\|^2 + V(q_1, \dots, q_n)$$

Suppose Σ_e is compact and the flow is ergodic on Σ_e . Then the time averages of $\frac{1}{2} m_j \|\dot{q}_j\|^2, j = 1, \dots, N$ are all equal.

Proof. Let $f_1 = \frac{1}{2} m_j \|\dot{q}_j\|^2$ and $f_2 = \frac{1}{2} m_k \|\dot{q}_k\|^2$ for given j, k . Define ϕ on P by interchanging q_j and q_k , likewise $\sqrt{m_j} \dot{q}_j$ and $\sqrt{m_k} \dot{q}_k$. Thus ϕ leaves E invariant, so maps Σ_e to Σ_e . The symplectic form here is

$$\omega_L = \sum m_i dq_i^\alpha \wedge d\dot{q}_i^\alpha$$

It follows easily that ϕ leaves the phase volume μ invariant (but not the symplectic form in general).

Since ϕ leaves μ and E invariant, it leaves μ_e invariant as well, and maps f_1 to f_2 . The result therefore follows from 3.7.26. ■

Note that this example also works on $P = T(M \times \dots \times M)$ where M is a Riemannian manifold.

The same argument can be used for the squares of individual components of the $\frac{1}{2} m_j \|\dot{q}_j\|^2$. The time average of each component is assigned a value $kT/2$ (k is a constant—Boltzmann's constant—whose numerical value depends on the choice of units). Thus the total time average of the kinetic energy is

$$\frac{3}{2} NkT$$

which implicitly defines the systems' temperature T . One can treat the pressure in an analogous manner; see 3.7.32.

One word of caution. We are tacitly assuming V is a C^∞ function. In the case that one wants to have particles enclosed in a container with hard walls, this smoothness question leads to nontrivial difficulties (c.f. Sect. 2.6).

(3) On R^{6N} consider

$$E(q, \dot{q}) = \sum_{j=1}^N \frac{1}{2} m_j \|\dot{q}^j\|^2 + \sum_{j=1}^N V_j(q_j) + \sum_{j < k} V_{jk}(q_j - q_k)$$

Here V_j represents an external potential (such as a container) and V_{jk} are interaction potentials between the particles. Assume again Σ_e is compact. Letting $X(x) = x$ in the virial theorem, we conclude that the time and space averages of the following are zero:

$$-2 \sum_{j=1}^N \frac{1}{2} m_j \|\dot{q}^j\|^2 + \sum_{j=1}^N \nabla V_j \cdot q_j + \sum_{j < k} [\nabla V_{jk}(q_j - q_k)] \cdot (q_j - q_k)$$

Let $3p|V|$ be the time average of the second term (p is the *pressure* and $|V|$ the volume of the "container"). Thus, if the flow is ergodic on Σ_e , we get the *equation of state*:

$$\frac{1}{3\mu_e(\Sigma_e)} p|V| = NkT - \left(\frac{1}{3}\right) \sum_{j < k} \int_{\Sigma_e} \nabla V_{jk}(q_j - q_k) \cdot (q_j - q_k) \mu_e$$

Note that on a curved space the pressure term would have to be modified by adding the "force" F_i to ∇V_i , as in Example 2.

Configuration space of N rigid points in $\mathbb{R}^3$

⚙ Configuration space of N rigid points in $\mathbb{R}^3$

Fix $d_{\{i,j\}} > 0$ for all $1 \leq i, j \leq N$. Then the configuration space of N rigid points in $\mathbb{R}^3$ with pairwise distances given by the fixed $\{d_{\{i,j\}}\}$ is

$$\{(x_1, \dots, x_N) \in (\mathbb{R}^3)^N \mid \forall 1 \leq i, j \leq N, \|x_i - x_j\|_{\mathbb{R}^3} = d_{\{i,j\}}\}$$

⚙ **Proposition:** For $N > 2$, the subspace of $(\mathbb{R}^3)^N$

⚙ Configuration space of N rigid points in $\mathbb{R}^3$

Fix $d_{\{i,j\}} > 0$ for all $1 \leq i, j \leq N$. Then the configuration space of N rigid points in $\mathbb{R}^3$ with pairwise distances given by the fixed $\{d_{\{i,j\}}\}$ is

$$\{(x_1, \dots, x_N) \in (\mathbb{R}^3)^N \mid \forall 1 \leq i, j \leq N, \|x_i - x_j\|_{\mathbb{R}^3} = d_{\{i,j\}}\}$$

is homeomorphic to

$$O(3, \mathbb{R}) \times \mathbb{R}^3$$

Riemannian mechanical manifold of a rigid body in $\mathbb{R}^3$ with a point fixed

Definition. Riemannian mechanical manifold of a rigid body in $\mathbb{R}^3$ with a point fixed $(SO(3, \mathbb{R}), m, \mathcal{F})$

Let m be a positive finite measure on $\mathbb{R}^3$ called *mass measure* not supported on any straight line through origin satisfying $\int_{\xi \in \mathbb{R}^3} \|\xi\|_{\mathbb{R}^3}^2 dm < \infty$.

Then consider the Riemannian metric on $SO(3, \mathbb{R})$ defined by

$$g_p^m(X, Y) := \int_{\xi \in \mathbb{R}^3} \langle X\xi, Y\xi \rangle_{\mathbb{R}^3} dm$$

for $p \in SO(3, \mathbb{R}), X, Y \in T_p SO(3, \mathbb{R}) \leq \text{Mat}(3, \mathbb{R})$ where

This defines

- a mechanical manifold

$$(SO(3, \mathbb{R}), g^m, \mathcal{F})$$

for an arbitrary *external force* $\mathcal{F}$ called **rigid body in $\mathbb{R}^3$ with a point fixed**

- $\mathcal{F} = 0$ then the Riemannian manifold

$$(SO(3, \mathbb{R}), g^m)$$

is called **Euler top**

We define the

- angular momentum**

$$\begin{aligned} L^m : TSO(3, \mathbb{R}) &\rightarrow \mathbb{R} \\ L^m(X_p) &:= \int_{\xi \in \mathbb{R}^3} p\xi \times X_p \xi dm \end{aligned}$$

- moment of inertia endomorphism**

$$\begin{aligned} I_m : \mathbb{R}^3 &\rightarrow \mathbb{R}^3 \\ I_m(v) &:= \int_{\xi \in \mathbb{R}^3} \xi \times (v \times \xi) dm \end{aligned}$$

- inertia ellipsoid**

$$\{\xi \in \mathbb{R}^3 \mid \langle I_m(\xi), \xi \rangle = 1\}$$

A **symmetry** of a rigid body is a m -invariant isometry $O \in O(3, \mathbb{R})$.

Proposition: There exists a (an unique?) linear isomorphism

$$\begin{aligned} \Gamma : \mathfrak{so}(3, \mathbb{R}) &\cong_{\text{LieAlg}_{\mathbb{R}}} (\mathbb{R}^3, \times) \\ A\xi &= \Gamma(A) \times \xi \end{aligned}$$

By this theorem we can convert all

on the $SO(3, \mathbb{R})$ side	on the $\mathbb{R}^3$ side
$A \in \mathfrak{so}(3, \mathbb{R})$ <i>rotation generator</i>	$\Omega = \Gamma(A) \in \mathbb{R}^3$ <i>axis of rotation</i>

on the $SO(3, \mathbb{R})$ side	on the $\mathbb{R}^3$ side
$\gamma : (a, b) \rightarrow SO(3, \mathbb{R})$ there is a $A : (a, b) \rightarrow \mathfrak{so}(3, \mathbb{R})$ such that $\dot{\gamma}(t) = \gamma(t)A(t)$	$\Omega(t) := \Gamma(A(t))$ such that $\dot{\gamma}(t)\xi = \gamma(t)A(t)\xi = \gamma(t)(\Omega(t) \times \xi)$
$g^m(X, Y)$	$\langle I_m \Omega_1, \Omega_2 \rangle$
$K(X) = \frac{1}{2}g^m(X, X)$ $K(\gamma A) = \frac{1}{2}g^m(\gamma A, \gamma A)$	$K = \frac{1}{2}\langle I_m(\Omega), \Omega \rangle$
<i>angular momentum</i> $L^m(\dot{\gamma}(t))$	$L^m(\dot{\gamma}(t)) = \gamma(t)I_m(\Omega(t))$
	<i>inertia ellipsoid</i> $\{\xi \in \mathbb{R}^3 \mid \langle I_m(\xi), \xi \rangle = 1\}$

Proposition: If γ is a geodesic of $(SO(3, \mathbb{R}), g^m)$ then $L^m(\dot{\gamma}(t))$ is a constant.

[1]

[2]

mass to moment

$$\begin{aligned} \text{mass measures} &\rightarrow \mathbf{Pd}(3, \mathbb{R}) \leftrightarrow \{\text{left invariant metics on } SO(3, \mathbb{R})\} \\ m &\mapsto I_m \end{aligned}$$

m	g^m	I_m
$\sum_{i=1}^N m_i \delta_i$	$\sum_{i=1}^N m_i \langle X\xi_i, Y\xi_i \rangle$	

Euler top

$L^m(\dot{\gamma}(t)) = \gamma(t)I_m(\Omega(t))$	$M(t) := I_m(\Omega(t))$
<i>Euler equations</i> $I_m(\dot{\Omega}) = I_m(\Omega) \times \Omega$	$\dot{M} = M \times \Omega$

1. https://www.math.tecnico.ulisboa.pt/~gcardoso/GeoRiem/nata_textb.pdf ↩

2. https://people.math.ethz.ch/~acannas/Student_Papers/BSc_Theses/2023_bsc_bachmann_llauger_geometric_mechanics.pdf ↩

IMPA Hidden symmetries in fluid dynamics 2024

#course/mini

In this series of lectures we discuss the recently discovered hidden (projective) symmetries in equations of fluid dynamics. This includes their theoretical foundations related to the algebraic structure of symmetry groups for both deterministic and statistical formulations. Then we consider their applications to the theory of hydrodynamic turbulence by showing how these symmetries underline the phenomena of Kolmogorov self-similarity and intermittency. We discuss several examples from rigorously solvable models to numerical challenges in fluid dynamics equations. We plan 8 lectures during 4 weeks in January 2024.

Wed & Fri, 10/1-2/2 at 15:00-17:00 in room 232.

VIDEOS on IMPA's YouTube Channel:

- [Lecture 1](#): Navier-Stokes (NS) and Euler equations. Symmetries. Conservation of energy.
- [Lecture 2](#): NS turbulence, Gaussian HIT, Gaussian vs. NS, viscous effects.
- [Lecture 3](#): Scaling anomaly, Random-h model, Multifractal theory.
- [Lecture 4](#): Solvable intermittent model of turbulence.
- [Lecture 5](#): Hidden scaling symmetry in shell models. Anomalous scaling laws.
- [Lecture 6](#): Kolmogorov multipliers. Rescaled NS equations.
- [Lecture 7](#): Hidden symmetries in NS turbulence.
- [Lecture 8](#): Group theory approach to hidden scaling symmetries.
- [Lecture 9](#): Group theory approach to quasi-Lagrangian formalism. Perspectives.

Notes: Simon ([lecture_1](#), [lecture_2](#), [lecture_3](#), [lecture_6_7](#)) & Alexei ([lecture_4_5_6](#), [lecture_8_9](#))

Boris Khesin - Geometric fluid dynamics

HENAN UNIVERSITY, Sept -- Oct 2021

Boris Khesin (University of Toronto)

Date/Time:

Beijing Time: Tue and Thu 8:30 -- 10am (Sept 14, 16, 21, 23, 28, 30, Oct 5 and 7)

Toronto Time (EST): Mon and Wed 8:30 -- 10pm (Sept 13, 15, 20, 22, 27, 29, Oct 4 and 6)

Zoom Meeting ID: 567 306 5241

Code: 123456

[Poster of the course](#)

Course description:

The course outlines group-theoretic, geometric, and topological approaches to hydrodynamics. We start by describing the Eulerian dynamics of an ideal fluid and the Korteweg--de Vries equation of shallow water from the group-theoretic and Hamiltonian points of view. We move on to cover the geometry of Casimirs for the Euler equation and helicity of vector fields. The Hamiltonian framework will also allow us to recover the motion of point vortices, vortex filaments and membranes. Finally, we will relate the differential geometry of diffeomorphism groups to problems of optimal mass transport.

References:

1. V.Arnold and B.Khesin: *Topological Methods in Hydrodynamics*, Appl. Math. Series, v. 125, Springer-Verlag 1998/1999, Second edition Springer-Nature 2021.
2. J.Marsden and T.Ratiu: *Introduction to Mechanics and Symmetry*, Texts in Applied Math., v. 17, Springer-Verlag, 1994/1999.
3. B.Khesin and R.Wendt: *The Geometry of Infinite-Dimensional Groups*, Ergebnisse der Math. und Grenzgebiete 3.Folge, v. 51, Springer-Verlag, 2009.

Tentative Plan of Lectures:

(slides and recordings will be linked to this page after each lecture)

I. Introducing the Euler equations. Its description as the geodesic flow.

[Slides](#); [Recording](#) (or [Recording from the Cloud](#), Access Passcode: mkRRP95#)

II. Equations on the dual Lie algebra, Lie-Poisson structures, Euler--Arnold equations.

[Slides](#); [Recording](#) (or [Recording from the Cloud](#), Access Passcode: 0l?ex4W&)

III. The Virasoro algebra and the KdV as an Euler equation.

[Slides](#); [Recording](#) (or [Recording from the Cloud](#), Access Passcode: 4FJ7H?#w)

IV. The Hamiltonian framework for hydrodynamics. Conservation laws for the Euler equations.

[Slides](#); [Recording](#) (or [Recording from the Cloud](#), Access Passcode: Ze0R7M%6)

V. Geometry of helicity.

[Slides](#); [Recording](#) (or [Recording from the Cloud](#), Access Passcode: FP@5!6yp)

VI. Point vortices and enstrophies in 2D.

[Slides](#); [Recording](#) (or [Recording from the Cloud](#), Access Passcode: @!KeZr?9)

VII. The Marsden--Weinstein symplectic structure on knots and vortex membranes.

[Slides](#); [Recording from the Cloud](#) (Access Passcode: P=F5@vxv)

VIII. Geometry of diffeomorphism groups and optimal mass transport.

[Slides](#); [Recording](#) (or [Recording from the Cloud](#), Access Passcode: r9FDLm&L)

**A talk for a general math audience:

"The mystery of pentagram maps"**

[Slides](#); [Recording](#) (or [Recording from the Cloud](#), Access Passcode: 04@Jq2NY)

Pierre Albin - Complex Algebraic Geometry

Math 514, Complex Algebraic Geometry (Fall 2020)

- [Home](#)
 - [Lecture Notes And Videos](#)
 - [Piazza](#)
 - [Moodle](#)
 - [Links](#)

Text

[Griffiths & Harris, *Principles of Algebraic Geometry*](#)

[Ballmann, *Lectures on Kähler Manifolds*](#)

[Voisin, *Hodge Theory and Complex Algebraic Geometry, I*](#)

Description

This course is an introduction to the geometry of Kähler manifolds and the Hodge structure of cohomology.

Kähler manifolds are at the intersection of complex analytic geometry, Riemannian geometry, and symplectic geometry. Moreover, every smooth projective variety is a Kähler manifold. All of this structure is reflected in a rich theory of geometric and topological invariants. In this course we will develop techniques from sheaf theory and linear elliptic theory to study the cohomology of Kähler manifolds.

In the setting of complex projective varieties one can use geometric and analytic methods to address questions in algebraic geometry. The results and examples then serve as guides in more algebraic approaches.

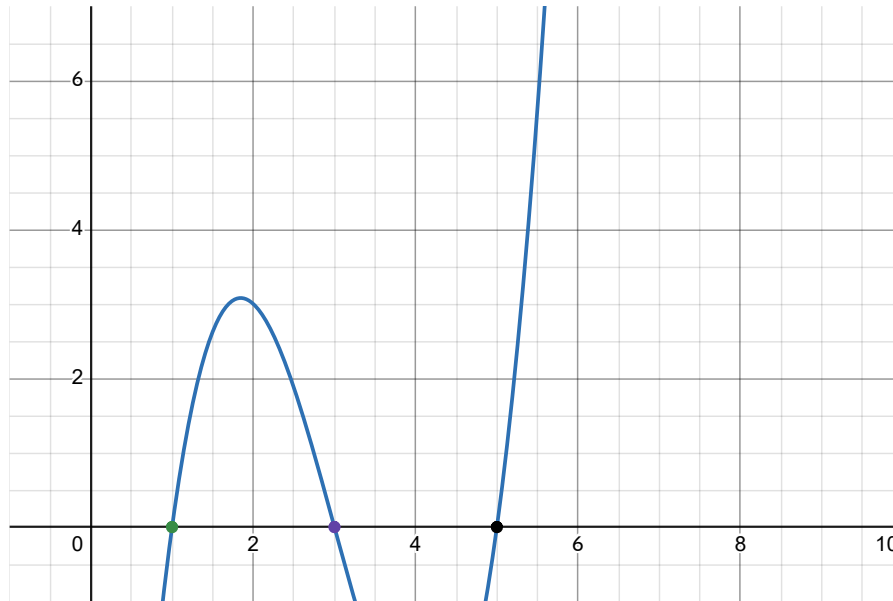
There is a lot of current research taking place in the setting of Kähler manifolds. For instance, very recently a couple of teams of researchers have managed to relate a notion of stability to the existence of Kähler-Einstein metrics on Fano spaces. For another example, one of the Clay millennium problems is to prove the Hodge conjecture: every Hodge class of a non-singular projective variety over $\mathbb{C}$ is a rational linear combination of cohomology classes of algebraic cycles. This course should be useful for students interested in research in geometry, be it differential or algebraic, and topology.

elliptic integrals

Let

$$p(X) = a_3 X^3 + a_2 X^2 + a_1 X + a_0 \in \mathbb{R}[X]$$

be a polynomial of degree 3 with three distinct real roots $\alpha_1 < \alpha_2 < \alpha_3$.



Then if

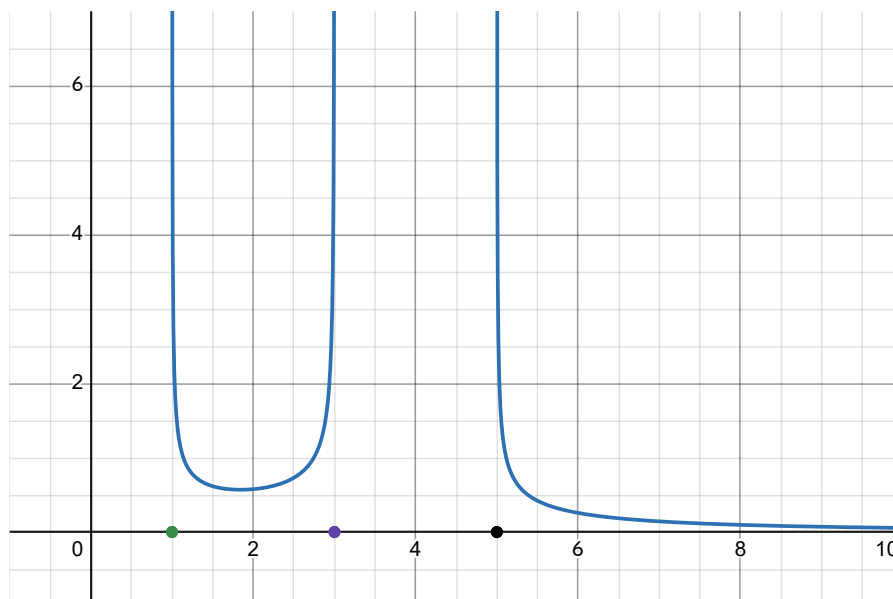
$$[a, b] \subset (-\infty, \alpha_1) \cup (\alpha_1, \alpha_2) \cup (\alpha_2, \alpha_3) \cup (\alpha_3, \infty)$$

such that

$$p([a, b]) \subset (0, \infty)$$

we may compute

$$\int_{[a,b]} \frac{1}{\sqrt{p(x)}} dx$$



Rather, on the elliptic curve

$$E_\alpha := \{(x, y) \in \mathbb{C}^2 \mid y^2 = p(x)\}$$

the integral of the 1-form

$$\int_{\delta} \frac{\mathrm{d} x}{y}=c_0 \int \frac{1}{\sqrt{p(x)}} \mathrm{d} x$$

for $\sigma \in H_1(E, \mathbb{Z})$

John Milnor - Singular Points of Complex Hypersurfaces

Chapter 1

Definition. Complex hypersurfaces

Let

$$f \in \mathbb{C}[z_1, z_2, \dots, z_{n+1}]$$

be a non-constant polynomial then

$$V(f)$$

is called **complex hypersurface**, an algebraic set consisting of the zeros of f .

A point $p \in V$ is a **regular point** of f if some $\frac{\partial f}{\partial z_i}$ does not vanish at p , making V a smooth $2n$ -manifold near p .

Let $p \in V(f)$ and we consider

$$K := V(f) \cap \partial B_\epsilon(p)$$

where $\partial B_\epsilon \subset \mathbb{C}^{n+1}$ is the $2n + 1$ -sphere of radius $\epsilon > 0$ centered at p .

If p is a regular point, and $\epsilon > 0$ small enough, the intersection

$$K := V(f) \cap \partial B_\epsilon(p)$$

is a smooth $2n - 1$ -manifold, embedded in

$$K \hookrightarrow \partial B_\epsilon(p) \cong S^{2n+1}$$

in an *unknotted manner*.

However,

Proposition: Consider the polynomial

$$f(z_1, z_2) := z_1^p + z_2^q$$

that has a critical point at $p = (0, 0)$. Assuming p, q are relatively prime integers and ≥ 2 we can conclude

$${}^1K := V(f) \cap \underbrace{\partial B_\epsilon(p)}_{S^3}$$

for small enough $\epsilon > 0$ is the torus knot of (p, q) type

$${}^1K \hookrightarrow S^3$$



$$K = \{z_1^p + z_2^q = 0\} \cap \{|z_1|^2 + |z_2|^2 = \epsilon\}$$

•

$$(ae^{iq\theta}, be^{ip\theta}) \in K$$

implies

$$a^p e^{iqp\theta} + b^q e^{ipq\theta} = 0, a^2 + b^2 = \epsilon$$

that is

$$b^2 = \epsilon - a^2 = (-a^p)^{2/q}$$

GSG on Rational curves and a conjecture of Drinfeld

✱ Abstract

Everyone is invited to the next talk of the Graduate Students' Group seminar, whose details are as follows:

Speaker: Dr. Sarbeswar Pal

Topic: Rational curves and an application of it to a conjecture of Drinfeld.

Abstract: In this lecture I will introduce the basic notions of rational curves in a smooth projective variety over complex numbers. We will see some interesting properties of the variety of rational curves and see how this helps us to prove a conjecture of Drinfeld.

Date: 15 March, 2025 (Tomorrow)

Time: 6:00 PM - 7:00 PM

Venue: AB2 2A or 2B

? Question

Let $X \subset \mathbb{C}P^M$ be a smooth projective variety. Does there exist a rational curve

$$\mathbb{C}P^1 \rightarrow X$$

In general, no.

TX is not **nef**.

$$c_1(TX) =$$

We know

$$\text{Mor}_d(P^1, X)$$

is a quasi-projective variety.

Suppose

$$f : P^1 \rightarrow X$$

is a rational curve and

$$TX \rightarrow X$$

is the projection from the tangent bundle.

Now the pullback

$$f^*TX \rightarrow P^1$$

is a vector bundle on P^1 .

Grothendieck. Any vector bundle on P^1 is a direct sum of line bundle

$$\bigoplus_i \mathcal{O}(a_i)$$

for $a_i \in \mathbb{Z}$

As

$$\dim X = n \implies TX \text{ has rank } n \implies f^*TX \text{ has rank } n$$

Then

$$f^*TX = \bigoplus_{i=1}^n \mathcal{O}(a_i)$$

for

$$a_1 \geq \dots \geq a_n$$

📌 **Definition.** f is **free** if $a_n \geq 0$ and is **non-free** otherwise

X is Fano if...

$$K_X := \Lambda^n TX$$

(canonical bundle)

Let L is a globally generated line bundle?

$$\Phi_L : X \rightarrow P(H^0(L)^*)$$

Then ...

📌 **Definition.** X is **Fano** if K_X **ample**.

Fact: $\exists$ an open set $X^{\text{free}} \subset X$ such that any rational curve which intersects X^{free} is **free**.

🔍 **Question**

is X^{free} empty? is it proper?

We don't know, in general.

📌 **Definition.** $X^{\text{nf}} := X \setminus X^{\text{free}}$

🔍 **Question**

X be a Fano projective manifold of (Picard rank 1) such that TX is not nef. Then X^{nf} is nonempty.

1. What can we say about $\dim X^{\text{nf}}$?

Guess: X^{nf} is pure of co-dim 1.

📖 **Example**

• P^n is Fano of Picard rank 1, and tangent bundle is...

• $X = \text{Gr}(r, \mathbb{C}^n)$

Examples of non-nef

Let X be a general hypersurface in P^n of degree $d \leq n$.

- Intersection of two quadratics
- X is curve

$$M(r, n)$$

be moduli of stable vec buns. It is smooth, Fano of Picard rank 1

$E \in M(r, n)$ is called **wobbly** if

$$\exists \phi : E \rightarrow E \otimes K_C$$

such that $\phi^m = 0 : E \rightarrow E \otimes K_C$.

Drinfeld's conjecture

Locus of wobbly bundles is pure of codim 1

 A vector bundle is wobbly if $\exists$ a non-free rational curve passing through E .

Conjecture

Let X be Fano, Picard rank 1, different from P^n . Then any non-constant endomorphism

$$X \rightarrow X$$

is isomorphism.

All proved examples (Grassmanians, homogeneous spaces?) have TX nef.

Guess

Let X Fano, Picard rank 1, $X \neq P^n$, TX is not nef

$$f : X \rightarrow X$$

then

$$f^{-1}(X^{\text{nf}}) = X^{\text{nf}}$$

This proves the conjecture.

March 17-19, 2025 - Geometric Aspects of Algebraic Varieties, IISER Mohali

GAAV 2025

GEOMETRIC ASPECTS OF ALGEBRAIC VARIETIES

MARCH 17-19
2025

IISER MOHALI

IN HONOR OF THE 65TH BIRTHDAY OF
PROF. KAPIL HARI
PARANJPE



LIST OF SPEAKERS

Nitin Nitsure, Indranil Biswas, C S Rajan, Dipendra prasad, S Ramanan, AJ Parameswaran, V Srinivas, Dinakar Ramkrishnan, Arvind Nair, Krishna Hanumanthu, Omprokash Das, A. Beauville, Claire Voisin, Madhav Nori

LINKS



<https://sites.google.com/view/gaav65>



gaav65@iisermohali.ac.in



THIS CONFERENCE IS BEING FUNDED BY NBHM AND IISER MOHALI

GEOMETRIC ASPECTS OF ALGEBRAIC VARIETIES

MARCH 17-19, 2025
09:30-17:00 LH4



Prof. Dipendra Prasad
19th March, 11:00-12:00



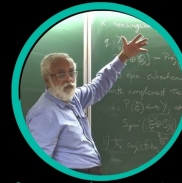
Prof. Indranil Biswas
17th March, 12:00-13:00



Prof. Nitin Nitsure
19th March, 14:30-15:30



Prof. Madhav Nori
19th March, 09:30-10:30



Prof. Vasudeva Srinivas
17th March, 11:00-12:00



Prof. Krishna Hanumanthu
17th March, 16:00-17:00



Prof. Claire Voisin
17th March, 14:30-15:30



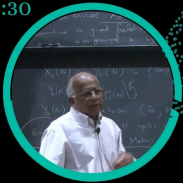
Prof. Arvind Nair
18th March, 09:30-10:30



Prof. Arnaud Beauville
18th March, 14:30-15:30



Prof. Omprokash Das
18th March, 11:00-12:00



Prof. Dinakar Ramakrishnan
19th March, 12:00-13:00



Prof. C S Rajan
18th March, 16:00-17:00



Prof. S Ramanan
19th March, 16:00-17:00



Prof. A J Parameswaran
18th March, 12:00-13:00

GAAV

IISER MOHALI



An event in honor of Professor Kapil Paranjape's 65th birthday.

Geometric Aspects of Algebraic Varieties

(Supported by NBHM and IISER Mohali)



MARCH
17-19
2025
IISER MOHALI

Conference
celebrating the
65'th Birthday of
Prof. Kapil Hari Paranjape

LIST OF SPEAKERS

Indranil Biswas, Arvind Nair
Nitin Nitsure, C S Rajan
Dinakar Ramkrishnan
A. J. Parameswaran
Dipendra Prasad
Omprokash Das
Arnaud Beauville*
Vasudevan Srinivas
Krishna Hanumanthu
Sundararaman Ramanan
Madhav Nori*, Claire Voisin*



GAAV
2025

VENUE

LH4, Lecture Hall Complex, IISER Mohali

*Online talk

17th March		Talks
9:30-10:30	Registration	
10:30-11:00	Tea	
11:00-12:00	Vasudevan Srinivas , University of Buffalo—SUNY (USA)	<i>Title: Some finiteness results for the étale fundamental group in positive characteristics</i> In this talk will discuss some results on étale fundamental groups of varieties over an algebraically closed field of characteristic $p > 0$, based on joint work with Helne Esnault and other coauthors. One result, along with Mark Schusterman, is that the tame fundamental group is finitely presented for such a variety which is the complement of an SNC divisor in a smooth projective variety. A second, along with Jakob Stix, is to give an obstruction for a smooth projective variety to admit a lifting to characteristic 0, in terms of the structure of its étale fundamental group as a profinite group. We will finally touch on some open questions.
12:00-13:00 	Prof. Indranil Biswas, Shiv Nadar University Delhi	<i>Title: A canonical connection on vector bundles on Riemann surfaces and Quillen connection on the theta bundle</i> We investigate the differential geometric aspects of some canonical torsors on the moduli space of vector bundles on a compact Riemann surface. (Joint work with Jacques Hurtubise.)
13:00-14:30	Lunch	
14:30-15:30	Prof. Claire Voisin, Jussieu, Paris	<i>Title: Universally defined cycles</i> I prove that universally defined cycles on smooth varieties of dimension d are uniquely given by polynomials in Chern classes. I prove a similar result for universally defined cycles on products of smooth varieties of dimension $d_1, \dots, d_n$. I will also discuss a related statement, which is still open, concerning universally defined cycles on powers of smooth varieties of a given dimension, and explain the motivation for this work.
15:30-16:00	Tea	

17th March		Talks
16:00-17:00	Prof. Krishna Hanumanthu, CMI	<i>Title: Some recent work on Seshadri constants</i> Let X be a projective variety, and let L be an ample line bundle on X. For a point x in X, the Seshadri constant of L at x is defined as the infimum, taken over all curves C passing through x, of the ratios $L.C/m$, where $L.C$ denotes the intersection product of L and C, and m is the multiplicity of C at x. This was introduced by J.-P. Demailly in 1990, inspired by Seshadri's ampleness criterion. These constants contain interesting and surprising information about the properties of the line bundle as well as the geometry of the variety. This notion has been extended in many directions, such as multi-point Seshadri constants and Seshadri constants for vector bundles of arbitrary rank. Seshadri constants are also related to certain well-known conjectures about linear system of plane curves. We will discuss some recent developments in this active area.

18th March		Talks
9:30-10:30	Prof. Arvind Nair	<i>D-modules on flag varieties and extensions in some categories of representations</i> I will show how to make some computations of groups in categories of representations of real reductive groups using D-modules/constructible sheaves of (enhanced) flag varieties, via the Beilinson-Bernstein localization theory. This leads to some curious numerical facts that I will try to motivate from arithmetic geometry.
10:30-11:00	Tea	
11:00-12:00	Prof. Omprokash Das	
12:00-13:00	Prof. A.J. Parameswaran	
13:00-14:30	Lunch	
14:30-15:30	Prof. Arnaud Beauville	
15:30-16:00	Tea	

18th March		Talks
16:00-17:00	C. S. Rajan , Ashoka University	<i>Title: Non-lacunarity of coefficients of linear combinations of L-functions of l-adic Galois representations</i> We extend results of Serre on non-lacunarity of prime and integral coefficients of L-functions attached to l-adic Galois representations to linear combinations of such L-functions. We apply them to Fourier coefficients of L-functions attached to modular forms and L-functions of induced representations. This is joint work with Rishabh Agnihotri and Mihir Sheth.

19th March		Talks
9:30-10:30	Prof. Madhav Nori	
10:30-11:00	Tea	
11:00-12:00	Prof. Dipendra Prasad	
12:00-13:00	Prof. Dinakar Ramakrishnan	
13:00-14:30	Lunch	
14:30-15:30	Prof. Nitin Nitsure	
15:30-16:00	Tea	
16:00-17:00	Prof. S. Ramanan	<i>Title: Quadric Geometry and Hyper-elliptic curves</i> The relationship between Quadric Geometry and hyper-elliptic curves has been much studied and goes back about 50 years - Indeed in 6 dimensions and genus 2, even 150 years! Recently I reworked the general case, particularly in order to make things work over a number field. I will talk about the modified approach and also refer to some recent results and open questions.

Prof. Indranil Biswas, Shiv Nadar University Delhi


Quote

During the early 90s, KHP used to talk about Hodge theory during lunch time.

- Prof. Indranil Biswas

[Jacques Hurtubise \(mathematician\)](#) - Wikipedia

[2102.00624] [A canonical connection on bundles on Riemann surfaces and Quillen connection on the theta bundle](#)

Let X with genus ≥ 2 be a compact RS and

$$E \rightarrow X$$

be degree 0 vector bundle.

E is **stable** if degree of every sub-bundle $F \leq E$ is non-zero

M moduli space of stable vec bun of rank r degree 0

NS: A stable E admits a unique unitary flat connection.

$$M = \frac{\mathrm{Hom}(\pi_1(X,u_0),U(r))}{U(r)}$$

$$T_E M = H$$

[1210.1643] [Rank one connections on abelian varieties, II](#)

Kabeer Manali Rahul - Metric techniques for triangulated categories

Quote

Speaker: Kabeer M. R. (Australian National University)

Time: Thursday, 20/3/25, 5PM.

Room: AB2-2A

Title : Introduction to metric techniques for (triangulated) categories

Abstract : Neeman has recently introduced certain techniques for triangulated categories which are analogous to notions related to a metric space. These techniques have been used to prove many interesting results in algebraic geometry, including three conjectures. In this talk, I will try to give some motivation, and an overview of these techniques. If time permits, I will also talk about some new representability theorems which have been proven using them.

#people Kabeer Manali Rahul

- MS Thesis: [DSpace@IISERMohali: Riemann Surfaces](#)
 - [210.212.36.82:8080/jspui/bitstream/123456789/1538/3/MS Thesis MS15152.pdf](#)
- [Kabeer Manali Rahul | kabeermr](#)
- [arxiv.org/pdf/1901.01453](#)
- [Triangulated category - Wikipedia](#)

examples of triangulated categories

Let $\mathcal{A}$ be an Abelian category (eg $\text{Mod}(R)$) and $\text{Ch}(\mathcal{A})$ be category of chain complexes

- $K(\mathcal{A})$
- $D(\mathcal{A})$ derived category

Given a Noetherian R

- $K^b(\text{proj-}R)$
- $D^b(\text{Mod}(R))$

Given a Noeth scheme X

- $\text{Perf}(X)$
- $D^b(\text{coh}(X))$

$(1) \cong (2) \iff X \text{ is regular} \iff R \text{ is ?regular?}$

keywords

- Morita theory

 **(Rieckerd)** Let R, S be Noeth rings then

Harish Seshadri - Comparison theorems in Kahler geometry

#talk/attended

Abstract

Dear All,

You are cordially invited to the following mathematics seminar.

Speaker: Prof Harish Seshadri (IISC Bangalore)

Title: Comparison theorems in Kahler geometry

Abstract: I will survey recent results on the volume, diameter, and the first eigenvalue of compact Kahler manifolds with positive curvature.

Room: AB2-2A

Time: Monday, 16/6/25 at 11AM.

Best regards.

Pranab Sardar.

GIT $\rightarrow$ moduli of vector bundles, stability of vector bundles $\rightarrow$ Einstein-Kähler metrics

🗨 [Literature recommendations in between Algebraic and Differential Geometry : r/math \(reddit.com\)](#)

On the more analytic side, people are interested in using algebraic+differential techniques to solve differential equations (canonical metrics on varieties, vector bundles) but usually the algebraic input is *fairly* limited (you sort of know it ahead of time, and then all the true difficulty is in analysis). The standard books here are Gabor's *Introduction to Extremal Kähler metrics* and Kobayashi *Differential geometry of complex vector bundles* to get started.

- Donaldson's recent work centers on a problem in complex differential geometry concerning a conjectural relationship between algebro-geometric "stability" conditions for smooth projective varieties and the existence of "extremal" Kähler metrics, typically those with constant scalar curvature (see for example [cscK metric](#)). ^[1]
 - Donaldson obtained results in the toric case of the problem (see for example [Donaldson \(2001\)](#)).
 - He then solved the Kähler–Einstein case of the problem in 2012, in collaboration with Chen and Sun. This latest spectacular achievement involved a number of difficult and technical papers.
 - The first of these was the paper of [Donaldson & Sun \(2014\)](#) on Gromov–Hausdorff limits.
 - The summary of the existence proof for Kähler–Einstein metrics appears in [Chen, Donaldson & Sun \(2014\)](#). Full details of the proofs appear in Chen, Donaldson, and Sun ([2015a](#), [2015b](#), [2015c](#)).

1. https://en.wikipedia.org/wiki/Simon_Donaldson#Research $\leftrightarrow$

Conjecture on Fano manifolds and Veblen Prize

See also: [K-stability](#) and [K-stability of Fano varieties](#)

In 2019, Donaldson was awarded the [Oswald Veblen Prize in Geometry](#), together with [Xiuxiong Chen](#) and [Song Sun](#), for proving a long-standing conjecture on [Fano manifolds](#), which states "that a Fano manifold admits a [Kähler–Einstein metric](#) if and only if it is [K-stable](#)". It had been one of the most actively investigated topics in geometry since its proposal in the 1980s by [Shing-Tung Yau](#) after he proved the [Calabi conjecture](#). It was later generalized by [Gang Tian](#) and Donaldson. The solution by Chen, Donaldson and Sun was published in the [Journal of the American Mathematical Society](#) in 2015 as a three-article series, "Kähler–Einstein metrics on Fano manifolds, I, II and III".¹²



#reference

- [Lectures on GEOMETRIC INVARIANT Theory & MODULI 1 | RADU LAZA](#)
- [\[1111.3032\] GIT and moduli with a twist](#)
- https://matematiflo.github.io/SoSe_2023/Vector_bundles_on_curves.pdf
- [THE LANDAU LECTURES 2014 - Prof. Gang Tian: Lecture 1](#)
- [Stability of algebraic varieties and Kahler geometry - Simon K. Donaldson](#)
- [Differential geometry of holomorphic vector bundles on a curve - Florent Schaffhauser](#)
- [LECTURES ON HERMITIAN-EINSTEIN METRICS FOR STABLE BUNDLES AND KÄHLER-EINSTEIN METRICS by Yum-Tong Siu](#)
- [Simon Donaldson: Kaehler-Einstein metrics and algebraic geometry I](#)
- [Simon Donaldson: Kaehler Einstein metrics and algebraic geometry II](#)
- [Simon Donaldson - The Ding functional, Berndtsson convexity and moment maps](#)
- [Simon Donaldson - Kahler Geometry](#)
- [Kahler-Einstein Metrics, Extremal Metrics and Stability](#)
- [Informal Talk on Kahler-Einstein Geometry, Pt. 1](#)
- [Informal Talk on Kahler-Einstein Geometry, Pt. 2](#)
- [Informal Talk on Kahler-Einstein Geometry, Pt. 3](#)
- [Informal Talk on Kahler-Einstein Geometry, Pt. 4](#)
- [Song Sun: Kahler-Einstein metrics](#) #ICBS2024
 - In this talk the speaker is represented to explain the paper X. Chen, S. Donaldson and S. Sun, "Kahler-Einstein metrics on Fano manifolds, I, II, III, JAMS 2015" which proves the equivalence between the existence of Kahler-Einstein metrics on a Fano manifold and K-stability. The key ingredient involves bridging differential geometry and algebraic geometry of singularity formations of Kahler manifolds. Time permitting we will also discuss some recent progress along this general direction by researchers in both the differential and algebro-geometric community.
- [S. Donaldson, Imperial College](#)
- [Notes on GIT and symplectic reduction for bundles and varieties - R.P. Thomas](#)

- <https://people.math.ethz.ch/~acannas/Papers/toric.pdf>
- <https://pi.math.cornell.edu/~tsh/cornell-only/cox-little-schenck-toric.pdf>
- [Toric Kahler geometry: October 5 – 9, 2015 – SCGP](#) and [videos](#)
- [tv-video-course.pdf](#)

Quote

Algebraic Geometry through Toric Varieties

Book: Toric Varieties, David A. Cox, Hal Schenck, and John B. Little.

Prerequisites: Proofs. Some abstract algebra including ring theory (e.g. MAT 313) is highly recommended.

Area: Algebraic Geometry.

Description: Fundamentally, algebraic geometry is the study of the spaces formed by looking at the zero sets of systems of polynomials. Such spaces are called varieties. Toric varieties are a particularly nice class of varieties that can be represented by a combinatorial object called a fan. By examining the fan, many important computations can be carried out for toric varieties with relative ease, while they are hard to do for general varieties.

In this project, our goal will be to understand the correspondence between toric varieties and fans. We will also see how to use the fan to compute important data such as the divisor class group and resolution of singularities for a toric variety.

This project does not assume prior knowledge of algebraic geometry, so we will most likely begin by reading some key sections from a text on undergraduate algebraic geometry before starting the main text (although the main text does review all necessary concepts from algebraic geometry).

PhD25 project

Quote

PSD - Dr Amaranta Membrillo Solis

Cohomological rigidity of toric spaces

Supervisor: [Dr Amaranta Membrillo Solis](#)

Project description:

The cohomological rigidity problem asks whether a topological space is uniquely determined by its cohomology ring. The proposed project investigates this problem in the setting of toric spaces, which arise in toric topology, symplectic geometry, and mathematical physics. These spaces are highly symmetric and governed by combinatorial structures such as polytopes and simplicial complexes, making them an ideal testing ground for rigidity phenomena.

The central goal is to determine when and how cohomological data can classify manifolds arising as toric spaces, up to homotopy equivalence, homeomorphism or diffeomorphism. This involves constructing families of polytopes or simplicial complexes along with their associated toric spaces, developing computational tools to analyse their invariants, and identifying both confirming examples and counterexamples. By integrating methods from algebraic and geometric topology, the project seeks to advance our understanding of the structure and classification of manifolds equipped with an effective torus action.

Keywords: *algebraic topology, geometric topology, computational topology, combinatorics.*

Area: Pure mathematics

Funding Notes

1. Funded by QMUL S&E New Talent Research Enabling Scheme => 3 years stipend and Home tuition fees.

2.n.b. students with International fee status are required to cover the difference between Overseas and Home tuition.

3.Stipend: Annual tax-free salary (e.g. £21,237 for 2024-2025 academic year; TBC for 2025-2026).

If you wish to inquire about the project, please contact the supervisor Dr Amaranta Membrillo Solis (i.a.membrillosolis@qmul.ac.uk).

[How to apply](#)

[Entry requirements](#)

[Fees and funding](#)

Nicolai Reshetikhin - Integrable systems and representation theory: geometry, algebra and analysis

✧ Abstract

This talk is an overview of some of the research directions that I am currently interested in. We will start with basic notions. The notion of an integrable system is clearly defined in the Hamiltonian mechanics, it extends naturally to quantum systems and to models of statistical mechanics where the transfer matrix is considered as a quantum evolution. After this introduction, I will focus on some examples that clearly illustrate the relation between integrable systems and representation theory. If time permit, I will also mention another research direction which is very important for quantum field theory but is not a part of the "world of integrable systems". It is the quantization of gauge theories.

Professor Nicolai Reshetikhin was born in Leningrad, former Soviet Union, now St. Petersburg, Russia. In 1982, he graduated from Leningrad State University with a bachelor's degree and a master's degree. In 1984, he graduated from the Steklov Institute of Mathematics and obtained a PhD degree. He has taught at well-known universities such as Harvard University, the University of California, Berkeley and the University of Amsterdam. He was invited twice to give a talk at the ICM International Conference of Mathematicians, one of which was a plenary talk. Professor Reshetikhin's main research interests include quantum topology, quantum groups and their representations, classical and quantum integrable systems, and integrable models statistical mechanics. He is one of the founders of quantum group theory, one of the authors of Reshetikhin-Turaev invariant, has important results in the theory of quantum integrable system, in Poisson and symplectic geometry, in the theory of quantum Kac Moody algebra. In 2010, he was elected as a Foreign member of the Royal Danish Academy. In 2021, he became a Fellow of the American Mathematical Society.

A Hamiltonian system in $SU(n)$

We have (almost) coordinates on $SU(n)$

$$g_{ij}$$

Cotangent bundle to $SU(n)$

$$T^*SU(n) \cong \mathfrak{su}(n, \mathbb{C})^* \times SU(n)$$

with symplectic structure

$$\omega = \sum_{ij} d\Pi_{ij} \wedge dg_{ij}$$

where $\sum_{i=1}^n \Pi_{ii} = 0$.

Let $X \in \mathfrak{su}(n, \mathbb{C})^*$

$$H_k := \text{tr}(X^k)$$

Proposition:

$$\{H_k, H_l\} = 0$$

The Hamiltonian vector fields can be solved explicitly

$$\begin{aligned} X(t) &= X \\ g(t) &= \exp \left(\sum_{k=2}^n t_k X^{k-1} \right) g \end{aligned}$$

projection method

Consider the conjugation action

$$\begin{aligned} SU(n) &\curvearrowright T^*SU(n) \\ (X, h) &\stackrel{h}{\mapsto} (hXh^{-1}, hgh^{-1}) \end{aligned}$$

This action is Hamiltonian with moment map

$$\begin{aligned} \mu : T^*SU(n) &\rightarrow \mathfrak{su}(n)^* \\ (X, h) &\mapsto X - h^{-1}Xh \end{aligned}$$

Consider a $SU(n)$ -orbit

$$\mathcal{O}_X = \{gXg^{-1} \mid g \in SU(n)\} \subset \mathfrak{su}(n)^*$$

Proposition:

$$\mu^{-1}(\mathcal{O}_X) \subset T^*SU(n)$$

is invariant under the flow

$$\mathcal{S}(\mathcal{O}) = \frac{\mu^{-1}(\mathcal{O})}{SU(n)}$$

is a *stratified* symplectic space, the Hamiltonian reduced space

The KdV equation

[1]

KdV	$u_t - 6uu_x + u_{xxx} = 0$
mKdV	$v_t - 6v^2v_xv_{xxx} = 0$
Schrodinger eigenfunction equation with potential V	$\psi_{xx} + (\lambda - V)\psi = 0$

Proposition: If v solves mKdV then

$$u := v^2 + v_x \text{ at } (x, t)$$

solves the KdV equation.



$$\begin{aligned} u &= v^2 + v_x \\ u_t &= 2vv_t + v_{tx} \\ u_x &= 2vv_x + v_{xx} \\ u_{xx} &= 2(v_x)^2 + 2vv_{xx} + v_{xxx} \\ u_{xxx} &= 4v_xv_{xx} + 2v_xv_{xx} + 2vv_{xxx} + v_{xxxx} \end{aligned}$$

•

$$u_t - 6uu_x + u_{xxx} = 2vv_t + v_{tx} - 6(v^2 + v_x)(2vv_x + v_{xx}) +$$

Let us fix $t_0 \in \mathbb{R}$ and get

$$\psi(x, t_0)$$

such that

$$v = \frac{\psi_x}{\psi} \text{ at } (x, t_0)$$

in

$$u = v^2 + v_x \text{ at } (x, t_0)$$

produces

$$\begin{aligned} u &= \left(\frac{\psi_x}{\psi}\right)^2 + \frac{\psi_{xx}}{\psi} - \frac{(\psi_x)^2}{\psi^2} \\ u &= \frac{\psi_{xx}}{\psi} \\ \psi_{xx} &= u\psi \end{aligned}$$

that is

$$\psi_{xx}(x, t_0) - u(x, t_0)\psi(x, t_0) = 0 \text{ for all } t_0 \in \mathbb{R}$$

which is a linear equation.

Say we know

$$u(x,0)$$

Then

$$u_2(x,t) := u(x+6\lambda t,t) + \lambda$$

also solves the KdV equation we obtain

$$\psi_{xx} + (\lambda - u)\psi = 0$$

direct scattering: Schrodinger eigenfunction equation with potential $u(x,t)$

Let $u(x,0)$ have compact support. Then eigenfunctions that satisfy

$$-\psi_{xx} + u(x,0)\psi = \lambda\psi$$

have

- bound states/discrete eigenvalues** $\lambda = -\kappa_n^2$ for $n \in \{1, \dots, N\}$

$$\psi_n(x,0) = N_n(0) \exp(-\kappa_n x)$$

- continuum of positive eigenvalues** $\lambda = k^2$

$$\psi_k(x,0) \sim \begin{cases} \underbrace{T(k,0)}_{\text{transmission}} \exp(-ikx) & \text{as } x \rightarrow -\infty \\ \exp(-ikx) + \underbrace{R(k,0)}_{\text{reflection}} \exp(ikx) & \text{as } x \rightarrow \infty \end{cases}$$

The scattering data at $t = 0$ is

$$\left((\kappa_1, N_1), \dots, (\kappa_N, N_N), R(k,0), T(k,0) \right)$$

evolution of scattering data

If

$$L : \mathcal{C}^\infty(M \times I) \rightarrow \mathcal{C}^\infty(M \times I)$$

is thought of as

$$L(t) : \mathcal{C}^\infty(M \times I) \rightarrow \mathcal{C}^\infty(M)$$

then

$$\frac{\partial}{\partial t} L\psi = \frac{dL}{dt} \psi + L \frac{\partial \psi}{\partial t}$$

$$L := -6\partial_x^2 - u(x,t)$$

$$M := -4\partial_x^3 - u\partial_x - \frac{u_x}{2}$$

Then

$$\begin{aligned} \frac{\partial}{\partial t} L\psi &= \frac{\partial}{\partial t} (-6\partial_x^2 \psi - u\psi) = -u_t \psi + L \frac{\partial \psi}{\partial t} \\ &\implies \frac{dL}{dt} = -u_t \end{aligned}$$

$$\begin{aligned}
[L, M] &= LM\psi - ML\psi \\
&= (-6\partial_x^2 - u) \left(-4\partial_x^3\psi - u\partial_x\psi - \frac{u_x}{2}\psi \right) \\
&\quad - \left(-4\partial_x^3 - u\partial_x - \frac{u_x}{2} \right) (-6\partial_x^2\psi - u\psi) \\
&= (-6\partial_x^2) \left(\cancel{-4\partial_x^3\psi}^0 - u\partial_x\psi - \frac{u_x}{2}\psi \right) \\
&\quad + (-u) \left(-4\partial_x^3\psi - u\partial_x\psi - \cancel{\frac{u_x}{2}\psi}^0 \right) \\
&\quad - (-4\partial_x^3) \left(\cancel{-6\partial_x^2\psi}^0 - u\psi \right) \\
&\quad - (-u\partial_x) (-6\partial_x^2\psi - u\psi) \\
&\quad - \left(-\frac{u_x}{2} \right) (-6\partial_x^2\psi - \cancel{u\psi}^0) \\
&= -uu_x - u_{xxx}
\end{aligned}$$

☰ Let M, L be self adjoint operators and $\lambda \in \mathbb{R}$. If

$$u_t - \mathcal{N}(u) = 0$$

can be expressed as

$$L_t + [L, M] = 0$$

and if

$$L\psi = \lambda\psi$$

then $\lambda_t = 0$ and $\psi_t = M\psi$ for $t > 0$.



$$L\psi(x, t) = \lambda(t)\psi(x, t)$$

differentiated with respect to t

$$L_t\psi + L\psi_t = \lambda_t\psi + \lambda\psi_t$$

•

$$\begin{aligned}
\lambda_t\psi &= (L - \lambda)\psi_t + [M, L]\psi \\
&= (L - \lambda)\psi_t + \lambda M\psi - LM\psi \\
&= (L - \lambda)(\psi_t - M\psi)
\end{aligned}$$

• Taking inner product

$$\begin{aligned}
\langle \psi, \psi \rangle \lambda_t &= \langle \psi, (L - \lambda)(\psi_t - M\psi) \rangle \\
&= \langle (L - \lambda)\psi, \psi_t - M\psi \rangle \\
&= 0
\end{aligned}$$

•

BIMSA - Integrable Systems and Algebraic Geometry - Beijing Summer Workshop in Mathematics and Mathematical Physics (June 24 – July 5, 2024)

[#talk/school/online](#)

✓ <https://web.archive.org/web/20240816054959/https://www.bimsa.cn/bmpsw/> >

```
frame: PDF
style: height: 900px;
urlSuffix: web.archive.org/web/20240816054959/https://www.bimsa.cn/bmpsw/
```

✓ <https://www.garyhu.me/static/pdf/BMPSW.pdf> ✓

```
frame: PDF
style: height: 900px;
urlSuffix: www.garyhu.me/static/pdf/BMPSW.pdf
```

✓ https://www.youtube.com/playlist?list=PLLGkFbxve670NmoPM-8D-0dvz8bKIP_IA ✓

```
frame: PDF
style: height: 900px;
urlSuffix: www.youtube.com/playlist?list=PLLGkFbxve670NmoPM-8D-0dvz8bKIP_IA
```

Introduction

This series of summer workshops is organized by the [Beijing Institute of Mathematical Sciences and Applications \(BIMSA\)](#). It aims at introducing young researchers to some of the active research areas in Mathematics and Mathematical Physics via a series of short lecture courses taught by some of the world's best mathematicians, combined with research talks given by world-renowned experts in the field. The theme for the inaugural workshop in Summer 2024 is [Integrable Systems and Algebraic Geometry](#).

Lecturers

- Pavel Etingof (Massachusetts Institute of Technology, USA)
- Samuel Grushevsky (Simons Center for Geometry and Physics, Stony Brook University, USA)
- Nikita Nekrasov (Simons Center for Geometry and Physics, Stony Brook University, USA)
- Andrei Okounkov (Columbia University, USA)
- Pavel Etingof (Massachusetts Institute of Technology, USA)

Title: The Hitchin System and its Quantization

(Lecture 1: [PDF](#) and [Youtube Video](#), Lecture 2: [PDF](#) and [Youtube Video](#), Lecture 3: [PDF](#) and [Youtube Video](#), Lecture 4: [PDF](#) and [Youtube Video](#), Lecture 5: [PDF](#) (preliminary) and [Youtube Video](#))

Abstract: Let G be a simple complex Lie group. I will review the classical Hitchin integrable system on the cotangent bundle to the moduli space $\text{Bun}_G(X)$ of principal G -bundles on a smooth complex projective curve X (possibly with punctures), as well as its quantization by Beilinson and Drinfeld using the loop group LG . I will explain how this system unifies many important integrable systems, such as Toda, Calogero-Moser, and Gaudin systems. Then I'll

discuss opers (for the dual group G_v), which parametrize the (algebraic) spectrum of the quantum Hitchin system. Finally, I will discuss the analytic problem of defining and computing the spectrum of the quantum Hitchin system on the Hilbert space $L^2(\text{Bun}_G(X))$, and will show that (modulo some conjectures, known in genus 0 and 1) this spectrum is discrete and parametrized by opers with real monodromy. Moreover, we will see that the quantum Hitchin system commutes with certain mutually commuting compact integral operators $H_{L,V}$ called Hecke operators (depending on a point $x \in X$ and a representation V of G_v), whose eigenvalues on the quantum Hitchin eigenfunction Ψ_L corresponding to a real oper L are real analytic solutions $\beta(x,x)$ of certain differential equations $D\beta=0$, $D^2\beta=0$ associated to L and V . This constitutes the analytic Langlands correspondence, developed in my papers with E. Frenkel and Kazhdan following previous work by Braverman-Kazhdan, Kontsevich, Langlands, Nekrasov, Teschner, and others. I will review the analytic Langlands correspondence and explain how it is connected with arithmetic and geometric Langlands correspondence.

- Samuel Grushevsky (Simons Center for Geometry and Physics, Stony Brook University, USA)

Title: [Integrable systems approach to the Schottky problem and related questions](#)

(Lecture 1: [Youtube Video](#), Lecture 2: [Youtube Video](#), Lecture 3: [Youtube Video](#), Lecture 4: [Youtube Video](#), Lecture 5: [Youtube Video](#))

Abstract: We will review the integrable systems approach to the classical Schottky problem of characterizing Jacobians of Riemann surfaces among all principally polarized complex abelian varieties. Starting with the Krichever's construction of the spectral curve from a pair of commuting differential operators, we will proceed to show that theta functions of Jacobians satisfy the KP hierarchy, and will review Novikov's conjecture (proven by Shiota) solving the Schottky problem by the KP equation. We will finally discuss some of the motivation for Krichever's proof of Welters' trisecant conjecture, and related characterizations for Prym varieties.

- [Nikita Nekrasov](#) (Simons Center for Geometry and Physics, Stony Brook University, USA)

Title: Integrable many-body systems and gauge theories

(Lecture 1: [Youtube Video](#), Lecture 2: [Youtube Video](#), Lecture 3: [Youtube Video](#), Lecture 4: [Youtube Video](#), Lecture 5: [Youtube Video](#)).

Abstract: Elliptic Calogero-Moser and Toda systems, Gaudin and other spin chains are algebraic integrable systems which have intimate connections to gauge theories in two, three, and four dimensions. I will explain two such connections: first, classical, through Hamiltonian reduction and second, quantum, through dualities of supersymmetric gauge theories.

- Andrei Okounkov (Columbia University, USA)

Title: From elliptic genera to elliptic quantum groups

(Lecture 1: [Youtube Video](#), Lecture 2: [Youtube Video](#), Lecture 3: [Youtube Video](#), Lecture 4: [Youtube Video](#), Lecture 5: [Youtube Video](#)).

Abstract: This course will be an example-based introduction to elliptic cohomology, Krichever elliptic genera, rigidity, and related topics. We will work our way towards the geometric construction of elliptic quantum groups.

Problem Sessions: [Problems for the course](#)

You can present your solutions either in person during the discussion session or by sending a pdf file to the course discussion groups in WeChat and in Telegram:

- Telegram: <https://t.me/+0w-1UHGT9bk0Njcx>

Invited Speakers (* = online talk)

- Mikhail Bershtein (The University of Edinburgh, UK)
- Alexander Bobenko* (Technische Universität Berlin, Germany)
- Alexei Borodin (Massachusetts Institute of Technology, USA)
- Thomas Bothner* (University of Bristol, UK)
- Alexander Braverman (University of Toronto, Canada)
- Ivan Cherednik (University of North Carolina, USA)
- Anton Dzhamay (BIMSA, China and The University of Northern Colorado, USA)

- Sergei Lando (Higher School of Economics and Krichever Center for Advanced Studies, Russia)
- Henry Liu (Kavli IPMU, Japan)
- Andrei Marshakov (Krichever Center for Advanced Studies, Russia)
- Grigori Olshanski* (Krichever Center for Advanced Studies and HSE, Russia)
- Senya Shlosman (Krichever Center for Advanced Studies, Russia and BIMSAs, China)
- Stanislav Smirnov (Geneva University, Switzerland and Krichever Center for Advanced Studies and St Petersburg State University, Russia)
- Alexander Veselov (Loughborough University, UK)
- Paul Wiegmann (University of Chicago, USA and BIMSAs, China)
- Anton Zabrodin (Krichever Center for Advanced Studies, Russia)
- Da-jun Zhang (Shanghai University, China)
- Youjin Zhang (Tsinghua University, China)

2024 Organizing Committee

- Anton Dzhamay (BIMSAs, China and The University of Northern Colorado, USA)
- Lynn Heller (BIMSAs, China)
- Andrei Marshakov (Krichever Center for Advanced Studies, Russia)
- Alexandra Skripchenko (Higher School of Economics, Russia)
- Andrei Okounkov (Columbia University, USA)
- Nicolai Reshetikhin (YMSC, Tsinghua University and BIMSAs, China)

Workshop Series Advisory Board

- Andrei Okounkov (Columbia University, USA)
- Nicolai Reshetikhin (YMSC, Tsinghua University and BIMSAs, China)
- Peng Shan (YMSC, Tsinghua University, China)
- Shing-Tung Yau (YMSC, Tsinghua University and BIMSAs, China)

Administrative Support

- Guangqiang Tie (BIMSAs, China)
- Xinyao Meng (BIMSAs, China)
- Fan Liu (BIMSAs, China)

Technical (Zoom) Support

- Ivan Sechin (BIMSAs, China)

Conference Poster

The poster can be downloaded [here](#).

Integrable systems approach to the Schottky problem and related questions

Samuel Grushevsky (Simons Center for Geometry and Physics, Stony Brook University, USA)

Summary

We will review the integrable systems approach to the classical Schottky problem of characterizing Jacobians of Riemann surfaces among all principally polarized complex abelian varieties.

- Starting with the Krichever's construction of the spectral curve from a pair of commuting differential operators,
- we will proceed to show that theta functions of Jacobians satisfy the KP hierarchy,
- and will review Novikov's conjecture (proven by Shiota) solving the Schottky problem by the KP equation.
- We will finally discuss some of the motivation for Krichever's proof of Welters' trisecant conjecture,
- and related characterizations for Prym varieties.

$$\mathcal{J}(M_g) \hookrightarrow \mathcal{A}_g \text{ moduli space}$$

$$C \mapsto \text{Jac}(C)$$

Schottky problem: describe $\mathcal{J}(M_g) \subset \mathcal{A}_g$
 (weaker version: given $C \hookrightarrow A$ is $A = \text{Jac}(C)$?)
 (weaker version: given $C \xrightarrow{\text{curve}} A \xleftarrow{\text{abelian variety}}$ is $A = \text{Jac}(C)$?)

Keystone: solve differential equations via functions originating from curves, and these diff. eq. will solve the Schottky problem

commuting diff operators

A general diff operator in one variable $x \in \mathbb{C}$

$$L = \sum_i u \frac{d^i}{dx^i}$$

want eigenfunctions of L

$$L|\psi\rangle = (\text{const})|\psi\rangle$$

Show that it is enough to consider

$$L = \frac{d^n}{dx^n} + \sum_i u_i \frac{d^i}{dx^i}$$

≡ Consider

$$L = \frac{d^n}{dx^n} + \sum_i u_i \frac{d^i}{dx^i}$$

then for all $x_0 \in \mathbb{C}$, $\exists!$ formal solution $|\psi\rangle$ of

$$L|\psi\rangle = k^n |\psi\rangle$$

of the form

$$\psi(x) = \sum_{s=0}^{\infty} z_s(x) k^{-s} e^{k(x-x_0)}$$

such that $z_0(x_0) = 1$ and for all s $z_s(x_0) = 1$

So at each order k^{-s} , solve for the next ≈ 5

Exercise. do this completely correctly by yourself $\square$

Corollary Any (formal) solution of $L\psi = k^n \psi$ has the form $\psi(x, k) = \psi(x, k, x_0) \cdot A(k, x_0)$
Exercise. prove this

Cor If $[L_1, L_2] = 0 = [L_1, L_3] = 0$

then $[L_2, L_3] = 0$

Proof $\forall \psi(x, k; x_0)$ [eigenfunction for L_1]

$$L_2 \psi(x, k, x_0) = A_2(k) \psi(x, k, x_0) \Rightarrow [L_2, L_3] \psi(x, k, x_0) = 0$$

$$L_3 \psi(x, k, x_0) = A_3(k) \psi(x, k, x_0)$$

$$\Rightarrow [L_2, L_3] \text{ has } \infty \text{ kernel (for all } k, x_0) \Rightarrow [L_2, L_3] = 0 \quad \square$$

Then $L_2: \mathcal{L}(E) \rightarrow \mathcal{L}(E)$. Let $Q_E(\alpha)$ be the characteristic polynomial of L_2 . (for E finite-dim)

Claim: $Q_E(\alpha)$ depends polynomially on E

So $Q(\alpha, \beta) \in \mathbb{C}[\alpha, \beta]$

But then $Q(L_1, L_2) \Big|_{\mathcal{L}(E)} = 0 \quad \forall E$

$\Rightarrow Q(L_1, L_2) \equiv 0$

Centralizer of L_1 is fin. dim

Growth of polynomial ring in two variables

Exercise:

1) Show that if $[L_1, L_2] = 0$

$\Rightarrow L_2 = \frac{d^m}{dx^m} T$

2) $\dim L_2$ if $[L_1, L_2] = 0$
 $\leq m+1$

Loray - Painlevé equations and isomonodromic deformations

Summary

Abstract - In these lectures, we use the material of V. Heu and H. Reis' lectures to introduce and study Painlevé equations from the isomonodromic point of view. The main objects are rank 2 systems of linear differential equations on the Riemann sphere, or more generally, rank 2 connections. We will mainly focus on the case they have 4 simple poles, corresponding to the Painlevé VI equation, while other Painlevé equations correspond to confluence of these poles.

First, we settle the Riemann-Hilbert correspondance which establish, roughly speaking, a one-to-one correspondance between connections and their monodromy data, once the poles are fixed. This correspondance is analytic, but not algebraic, very transcendental. Then isomonodromic deformations arise when we deform poles and connection without deforming the monodromy representation. Although the deformation is also transcendental in general, the coefficients satisfy a non linear polynomial differential equation, namely the Painlevé VI equation. By constructing an universal isomonodromic deformation, we explain how Malgrange proved the Painlevé property for isomonodromic deformation equations: solutions admit analytic continuation (with poles) outside of a fixed singular set. At the end, we can describe the Okamoto space of initial conditions for Painlevé VI equation, as well as its non linear monodromy. This can be used to prove the irreducibility of Painlevé VI equation, i.e. the absence of special first integrals, and therefore the transcendence of the general solution.

[1]

[2]

space.f.Painlevé equations

Painlevé equations

Definition. Painlevé equations

		Hamiltonian (time dependent)	Symmetries
I (Painlevé)	$\frac{d^2y}{dt^2} = 6y^2 + t$		
II (Painlevé)	$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$	with $q := y$ $p := y' + y^2 + \frac{t}{2}$ we have $H(p, q, t) := \frac{p(p - 2q^2 - t)}{2 - bq}$	
III (Painlevé)	$\frac{d^2y}{dt^2} = \frac{1}{y} \left(\frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} +$		

		Hamiltonian (time dependent)	Symmetries
IV (Gambier)	$\frac{d^2y}{dt^2} = \frac{1}{2y} \left(\frac{dy}{dt} \right)^2 + \frac{3}{2}y^3 +$		
V (Gambier)	$\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1} \right) \left(\frac{dy}{dt} \right)^2 + \frac{(y-1)^2}{t^2} \left(\alpha y + \frac{\beta}{y} \right)$		
VI (R. Fuchs)	$\frac{d^2y}{dt^2} = \frac{1}{2} \left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t} \right) \left(\frac{dy}{dt} \right)^2 + \frac{y(y-1)(y-t)}{t^2(t-1)^2} \left\{ \alpha \frac{dy}{dt} + \beta \right\}$		

🔗 (Poincare, Fuchs) The differential equations of the form

$$P(x, y, y') = 0$$

for $P \in \mathbb{C}[X, Y, Z]$ whose local solutions

$$y(x)$$

are transcendental with **good** analytic continuation are Riccati equation

$$y' = a(x)y^2 + b(x)y + c(x)$$

and Weirstrass equation

$$(y')^2 = y^3 + ay + b$$

Here, "good" analytic continuation means:

📌
Definition. Painleve property

A differential equation

$$P(x, y, y') = 0$$

is said to have the **Painleve property** if exists a finite $Z \subset \mathbb{CP}^1$ such that any local solution

$$y(x)$$

that is defined at the neighborhood of $x_0 \in \mathbb{CP}^1 \setminus Z$ can be meromorphically continued along any path

$$\gamma : [0, 1] \rightarrow \mathbb{CP}^1 \setminus Z$$

💡 (sketch of proof)

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}$$

where $P, Q \in \mathbb{C}[X, Y]$

- This equation have foliations whose leaves are graphs of local solutions

- This is the foliation associated to

$$Qdy - Pdx = 0$$

or equivalently to the vector field

$$Q\frac{\partial}{\partial x} + P\frac{\partial}{\partial y}$$

•

1. <https://hal.science/medihal-02273534v1> ↩

2. <https://av.tib.eu/series/1587> ↩

Henryk Żołądek - The Monodromy Group

NCTS Short Course on Riemann Hilbert Method in Integrable Systems

NCTS Short Course

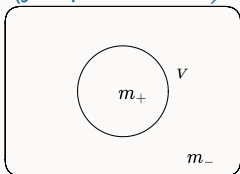
Riemann-Hilbert Method in Integrable Systems

Prof. Peter Miller (U. Michigan)

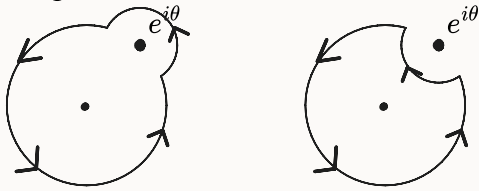
There will be 5 lectures. Each lecture takes about 2 hours, with outline as below.

Lecture I: Riemann-Hilbert Problems for Orthogonal Polynomials and Painlevé Equations

Abstract: This lecture will introduce the notion of Riemann-Hilbert problems in the theory of orthogonal polynomials and in the theory of Painlevé equations. We begin with orthogonal polynomials and develop the Fokas-Its'-Kitaev representation of them via a matrix Riemann-Hilbert problem. Then we discuss the representation of solutions of Painlevé equations via Riemann-Hilbert problems associated to the inverse monodromy problem for some linear differential equations. We develop isomonodromic Schlesinger transformations as a tool for generating families of solutions of Painlevé equations, focusing on the example of the rational solutions of Painlevé-II.

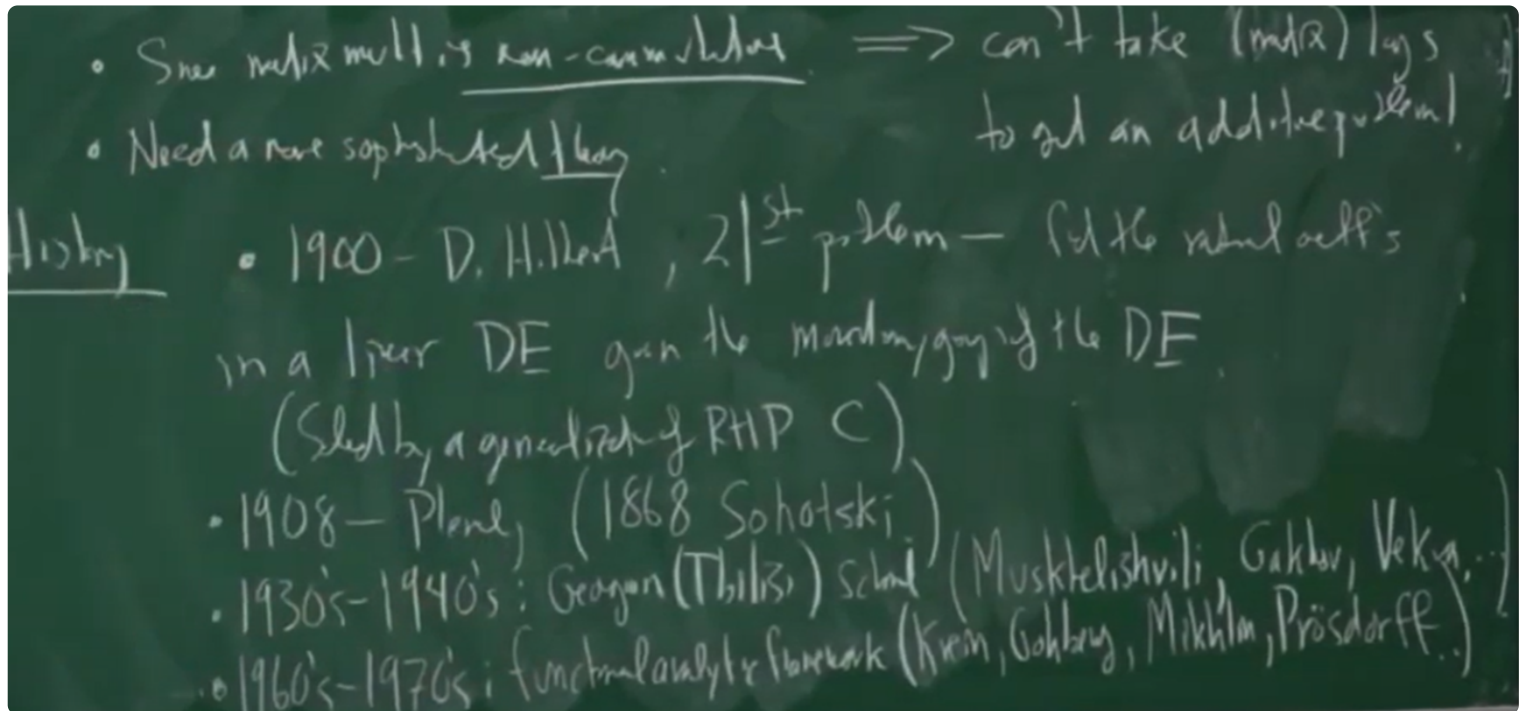
	RHP(A)	RHP(B)
given	smooth $V : S^1 \rightarrow \mathbb{C}$	smooth $V : S^1 \rightarrow \mathbb{C}$
find	functions m_+, m_- analytic inside and outside S^1 respectively such that $m_-(\infty) = 0$ and $m_+(e^{i\theta}) = m_-(e^{i\theta}) + V(e^{i\theta})$ (jump condition) 	functions m_+, m_- analytic inside and outside S^1 respectively such that $m_-(\infty) = 0$ and $m_+(e^{i\theta}) = m_-(e^{i\theta})V(e^{i\theta})$ (multiplicative jump condition)

	RHP(A)	RHP(B)
solution 1	Conditions imply $m_+(z) = \sum_{n \geq 0} a_n z^n$ convergent inside S^1 and $m_-(z) = \sum_{n \leq -1} b_n z^n$ with $V(e^{i\theta}) = \sum_{n \in \mathbb{Z}} v_n e^{in\theta}, \theta \in (-\pi, \pi]$ where the <i>jump condition</i> takes the form $\sum_{n \geq 0} a_n z^n = \sum_{n \leq -1} b_n z^n + \sum_{n \in \mathbb{Z}} v_n e^{in\theta}$ giving us $\begin{aligned} n \in \mathbb{N} &\implies a_n = v_n \\ n < 0 &\implies b_n = -v_n \end{aligned}$	Take logs! reduces the problem to RHP(A) $m_{\pm}(z) = \exp \left(\frac{1}{2\pi i} \int_{S^1} \frac{\log(V(w))dw}{w - z} \right)$ is smooth iff index of V ("winding number of V) is zero around S^1.
<i>Uniqueness</i>	Let $m_{\pm}, \tilde{m}_{\pm}$ be two solutions. Then $\Delta m_{\pm} := \tilde{m}_{\pm} - m_{\pm}$ by jump condition $\Delta m_+ = \Delta m_- \text{ on } S^1$ Then Δm is entire, and its bounded, so by <i>Louville</i> it has to be 0.	

	RHP(A)	RHP(B)
solution 2	$m(z) := \underbrace{\frac{1}{2\pi i} \int_{w \in S^1} \frac{V(w)dw}{w-z}}_{C_{S^1}[V](z)}$ which goes to 0 as $z \rightarrow \infty$ Integrate on these contours  then $m_-(e^{i\theta}) - m_+(e^{i\theta}) = \frac{1}{2\pi i} \int_{S^1_{e^{i\theta}}} \frac{V(w)}{w - e^{i\theta}}$ (Plemelj) which $= V(e^{i\theta})$ by residue theorem.	

History

[1]



80's : asymptotic inverse-scattering problems
(Beals, Coifman, Zhou)

90's : Asymptotic methods for RHPs with a large parameter
(Deift, Zhou)

- 1849
 - RHP(B) for $V(e^{i\theta}) = 1$ by B. Riemann in his thesis (where he cooked complex analysis and Riemann surfaces)
- 1900
 - D. Hilbert, 21st problem: Proof of the existence of linear differential equations having a prescribed monodromic group
- 1908
 - Plemelj solved the problem by a generalization of RHP(C)
 - 1868 solution?
- 1930s - 40s
 - Physics inspired problems
 - Georgian (Thilisi) school by Muskhelishvili, Gakhov, Vek
- 1960s-1970s
 - functional analytical framework by Kren, G
- 1980s
 - application to inverse scattering and integrable systems
 - generally for regular holonomic D-modules by [Masaki Kashiwara](#) (1980, 1984) and [Zoghman Mebkhout](#) (1980, 1984) independently
- 1990s
 - asymptotic methods with large parameters

RHP type C

- Given: $V : S^1 \rightarrow GL()$

orthogonal polynomials

Orthogonal polynomials.

$$\int_{\mathbb{R}} w(x)|x|^n dx < \infty, \quad \forall n = 0, 1, 2, 3, \dots$$
$$\langle p_j, p_k \rangle_w = \delta_{jk}, \quad \langle f, g \rangle_w := \int_{\mathbb{R}} f(x)g(x)w(x) \, dx.$$
$$\pi_n(x) := x^n - \sum_{k=0}^{n-1} \langle p_k, x \rangle_w p_k(x), \quad \gamma_n := \frac{1}{\sqrt{\langle \pi_n, \pi_n \rangle_w}}, \text{ and } p_n(x) := \gamma_n \pi_n(x).$$

algebraic study of ODEs



#reference

- [Workshop on Model Theory, Algebraic Dynamics, and Differential-Algebraic Geometry | Fields Institute for Research in Mathematical Sciences](#)

Let $P(X) \in \mathbb{C}[X]$ be a monic polynomial. We consider the ODE: looking for functions f such that

$$P(\mathfrak{D})f = 0$$

where $\mathfrak{D}$ is the derivative operator.

using primary decomposition for complex case

Now over the field $\mathbb{C}$, if we factorize the polynomial

$$P(X) = (X - \alpha_1)^{r_1} \dots (X - \alpha_k)^{r_k}$$

where r_k is the multiplicity of the root α_k then by using

☰ (General primary decomposition) Let T be an endomorphism on a vector space V and



$$p = p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}$$

be prime factorization of a monic polynomial p such that $p(T) = 0$. Then

$$V = \sum_{i=1}^k \ker p_i^{r_i}(T)$$

on the space of all solutions $V = \ker(P(D) : \mathcal{C}^n(\mathbb{R}, \mathbb{C}) \rightarrow \mathcal{C}^{n-1}(\mathbb{R}, \mathbb{C}))$ we have the operator

$$D : V \rightarrow V$$

which satisfies the polynomial $P(D) = 0$, so we get

$$V = \ker(D - \alpha_1)^{r_1} + \dots + \ker(D - \alpha_k)^{r_k}$$

So we must only solve the differential equation

$$(D - \alpha)^r f = 0$$

- **Base case:**

$$f' - \alpha f = e^{\alpha x} (e^{-\alpha x} f)'$$

- **Induction hypothesis:** We assume

$$(D - \alpha)^r f = e^{\alpha x} D^r (e^{-\alpha x} f)$$

holds for some r

- **Inductive step:**

$$\begin{aligned} (D - \alpha)^{r+1} f &= (D - \alpha) e^{\alpha x} D^r (e^{-\alpha x} f) \\ &= \alpha e^{\alpha x} D^r (e^{-\alpha x} f) + e^{\alpha x} D^{r+1} (e^{-\alpha x} f) - \alpha e^{\alpha x} D^r (e^{-\alpha x} f) \\ &= e^{\alpha x} D^{r+1} (e^{-\alpha x} f) \end{aligned}$$

- So the proposition holds for $r + 1$.
- Hence

$$(D - \alpha)^r f = 0 \iff D^r(e^{-\alpha x} f) = 0$$

which is true if and only if

$$e^{-\alpha x} f(x) = b_0 + b_1 x + \cdots + b_{r-1} x^{r-1}$$

- which means

$$\ker(D - \alpha)^r = \mathbb{C}\{e^{\alpha x}, x e^{\alpha x}, \dots, x^{r-1} e^{\alpha x}\}$$

Thus the functions

$$x^m e^{\alpha_i x}$$

for $0 \leq m \leq r_i - 1$ for all $1 \leq i \leq k$ form a basis for V , the space of all solutions of the differential equation.

In particular

$$\dim V = \deg p$$

Problem

Show that the cone over X

$$X \hookrightarrow \text{Cone}(X)$$

has the *homotopy extension property*.

Quote

Not all inclusions $A \subset X$ are created equal.

[1]

Definition. Homotopy extension property

Let X be a topological space and let $A \subset X$. We say (X, A) has the **homotopy extension property** if for a homotopy $f_\bullet : A \times I \rightarrow Y$ and a map $g_0 : X \rightarrow Y$ such that $f_0 = g_0|_A$ there exists a homotopy

$$g_\bullet : X \times I \rightarrow Y$$

that starts from $g_0 : X \rightarrow Y$ and *extends* f , that is,

$$f_t = g_t|_A, \forall t \in I$$

The property amounts to lifting $f_\bullet, g_0$ in

```
\documentclass{standalone}
\usepackage{tikz-cd}
\usepackage{amsmath}\usepackage{quiver}\usepackage{mathrsfs}

\begin{document}
\tikzcdset{scale cd/.style={every label/.append style={scale=#1},cells={nodes={scale=#1}}}}
% \begin{tikzcd}[scale cd=0.8]

%
https://q.uiver.app/#q=WzAsNCxbMCwxcLCJBIL0sWzAsMCwiWCJdLFsyLDAsIlkiXSxbMiwxcLCJZXntJfSJdLFswLDEsI
iIsMCx7InN0eWxlIjp7InRhaWwiOmsibmFtZSI6Imhvb2siLCJzaWRLIjoicG9wIn19fV0sWzEsMiwic197MH0iXSxbMCwzL
CJmX1xcYnVsbGV0IiwycXxbMywyLCJcXHBpXzAiLDJdXQ==
\begin{tikzcd}
X & \& Y \\
A & \& \{Y^I\} \\
\arrow["{g_0}", from=1-1, to=1-3]
\arrow[hook, from=2-1, to=1-1]
\arrow["{f_\bullet}", from=2-1, to=2-3]
\arrow["{\pi_0}", from=2-3, to=1-3]
\end{tikzcd}

\end{document}
```

to a $f_\bullet$ such that

```

\documentclass{standalone}
\usepackage{tikz-
cd}\usepackage{amsfonts}\usepackage{amsmath}\usepackage{quiver}\usepackage{mathrsfs}

\begin{document}
\tikzcdset{scale cd/.style={every label/.append style={scale=#1},cells={nodes={scale=#1}}}}
% \begin{tikzcd}[scale cd=0.8]

%
https://q.uiver.app/#q=WzAsNCxbMCwxLCJBIl0sWzAsMCwiWCJdLFsyLDAsIlkiXSxbMiwxcJZXntJfSJdLFswLDEsI
iIsMCx7InN0eWxlIjp7InRhaWwiOnsibmFtZSI6Imhvb2siLCJzaWRlIjoiaW9fV0sWzEsMiwz197MH0iXSxbMCwzL
CJmX1xcYnVsbGV0IiwyaXxbMywyLCJcXHBpXzAiLDJdLFsxLDMsImdfXfXidWxsZXQiLDdFdlFs2LDgsIiIsMix7InNob3J0Z
W4iOnsic291cmNlIjoyMCwidGFyZ2V0IjoyMH19XV0=
\begin{tikzcd}
X & \& Y & \backslash \\
A & \& \{Y^{\{I\}} \\
\arrow["{g_{\{0\}}}", from=1-1, to=1-3]
\arrow[""{name=0, anchor=center, inner sep=0}, "{g_{\bullet}}"{description}, from=1-1, to=2-3]
\arrow[hook, from=2-1, to=1-1]
\arrow[""{name=1, anchor=center, inner sep=0}, "{f_{\bullet}}", from=2-1, to=2-3]
\arrow["{\pi_0}", from=2-3, to=1-3]
\arrow[shorten <=2pt, shorten >=2pt, Rightarrow, from=1, to=0]
\end{tikzcd}

\end{document}

```

Now a cone

$$\text{Cone}(X) := \frac{X \times I}{X \times \{1\}}$$

$$X \cong X \times \{0\} \hookrightarrow \text{Cone}(X)$$

gives

[2]

1. math.jhu.edu/~jmb/note/hep.pdf ↩

2. [algebraic topology - Homotopy Extension Property of mapping cone - Mathematics Stack Exchange](#) ↩

reconstructing compact Hausdorff X from $\mathcal{C}(X, \mathbb{R})$

Let X be a compact Hausdorff space, $\mathcal{C}(X, \mathbb{R})$ be the ring of all continuous functions $X \rightarrow \mathbb{R}$ and

$$\max\mathrm{Spec} \mathcal{C}(X, \mathbb{R})$$

denote the maximal ideals in spectrum of $\mathcal{C}(X, \mathbb{R})$.

We have the surjective ring map

$$\mathrm{ev}_x : \mathcal{C}(X, \mathbb{R}) \rightarrow \mathbb{R}$$

whose kernel is a maximal ideal $\mathfrak{m}_x$.

Then

Proposition: Let X be a compact Hausdorff space. We have

$$\begin{aligned} X &\rightarrow \max\mathrm{Spec} \mathcal{C}(X) \\ x &\mapsto \mathfrak{m}_x \end{aligned}$$



and multiplying their values). For each $x \in X$, let $\mathfrak{m}_x$ be the set of all $f \in C(X)$ such that $f(x) = 0$. The ideal $\mathfrak{m}_x$ is maximal, because it is the kernel of the (surjective) homomorphism $C(X) \rightarrow \mathbb{R}$ which takes f to $f(x)$. If $\tilde{X}$ denotes $\text{Max}(C(X))$, we have therefore defined a mapping $\mu: X \rightarrow \tilde{X}$, namely $x \mapsto \mathfrak{m}_x$.

We shall show that μ is a homeomorphism of X onto $\tilde{X}$.

- i) Let $\mathfrak{m}$ be any maximal ideal of $C(X)$, and let $V = V(\mathfrak{m})$ be the set of common zeros of the functions in $\mathfrak{m}$: that is,

$$V = \{x \in X : f(x) = 0 \text{ for all } f \in \mathfrak{m}\}.$$

Suppose that V is empty. Then for each $x \in X$ there exists $f_x \in \mathfrak{m}$ such that $f_x(x) \neq 0$. Since f_x is continuous, there is an open neighborhood U_x of x in X on which f_x does not vanish. By compactness a finite number of the neighborhoods, say $U_{x_1}, \dots, U_{x_n}$, cover X . Let

$$f = f_{x_1}^2 + \dots + f_{x_n}^2.$$

Then f does not vanish at any point of X , hence is a unit in $C(X)$. But this contradicts $f \in \mathfrak{m}$, hence V is not empty.

Let x be a point of V . Then $\mathfrak{m} \subseteq \mathfrak{m}_x$, hence $\mathfrak{m} = \mathfrak{m}_x$ because $\mathfrak{m}$ is maximal. Hence μ is surjective.

- ii) By Urysohn's lemma (this is the only non-trivial fact required in the argument) the continuous functions separate the points of X . Hence $x \neq y \Rightarrow \mathfrak{m}_x \neq \mathfrak{m}_y$, and therefore μ is injective.

- iii) Let $f \in C(X)$; let

$$U_f = \{x \in X : f(x) \neq 0\}$$

and let

$$\tilde{U}_f = \{\mathfrak{m} \in \tilde{X} : f \notin \mathfrak{m}\}$$

Show that $\mu(U_f) = \tilde{U}_f$. The open sets U_f (resp. $\tilde{U}_f$) form a basis of the topology of X (resp. $\tilde{X}$) and therefore μ is a homeomorphism.

Thus X can be reconstructed from the ring of functions $C(X)$.

making $\mathbb{R}$ -algebras function spaces

- Let A be a commutative, associative $\mathbb{R}$ -algebra with 1_A .
- Let $|A| := \text{HomAlg}_{\mathbb{R}}(A, \mathbb{R})$ be all unital $\mathbb{R}$ -algebra homomorphisms.
 - Every element

$$x : A \rightarrow \mathbb{R}$$

defines a kernel which is either zero or maximal $\mathfrak{m}_x$

- $\tilde{A} := \{\text{ev}_f : |A| \rightarrow \mathbb{R} \text{ for some } f \in A\}$

is a commutative $\mathbb{R}$ -algebra

- These evaluation maps

$$\text{ev}_f : |A| \rightarrow \mathbb{R}$$

- There is a natural surjective homomorphism

$$\begin{aligned} A &\rightarrow \tilde{A} \\ f &\mapsto \text{ev}_f \end{aligned}$$

but it is not injective in general.

🔑 **Definition.** An $\mathbb{R}$ -algebra A is called **geometric** if

$$\bigcap_{x \in |A|} \ker x = 0$$

And

$$f \in \ker(A \rightarrow \tilde{A}) \iff \text{ev}_f = 0 \iff \forall x \in |A|, x(f) = 0 \iff f \in \bigcap_{x \in |A|} \ker x$$

thus

📖 Any geometric $\mathbb{R}$ -algebra A is canonically isomorphic to the $\mathbb{R}$ -algebra of functions on the dual space

$$|A|$$

We give the weak-* topology on $|A|$ such that every

$$f \equiv \text{ev}_f : M \rightarrow \mathbb{R}$$

in A is continuous.

Proposition: This makes $|A|$ a Hausdorff space.

Thus we understand

$$A \hookrightarrow \mathcal{C}(|A|, \mathbb{R})$$

Let A be a geometric $\mathbb{R}$ -algebra and $S \subset |A|$. Then the restriction

$$A(S) := \{f : S \rightarrow \mathbb{R} \mid \forall x \in S \exists \bar{f} \in A, U_x \subset S : f|_{U_x} = \bar{f}|_{U_x}\}$$

and we have the restriction

$$\text{res}_S : A \rightarrow A(S)$$

🔑 **Definition.** A geometric $\mathbb{R}$ -algebra is **complete** if the restriction

$$\text{res}_{|A|} : A \rightarrow A(|A|)$$

is **surjective**.

on $\mathcal{C}(X)$ for non-compact X

- $\text{maxSpec } \mathcal{C}(X)$ for non-cpt (nice enough) X are points of βX (Stone-Cech compactification)
<https://mathoverflow.net/a/3876>
- but $\beta\mathbb{R} \setminus \mathbb{R}$ (new points in compactified $\mathbb{R}$) is horribly large <https://math.stackexchange.com/a/1790467> and hard to describe (AoC blinds us); its supposed to be larger than one/two points because $\sin(x)$ is continuous on $\beta\mathbb{R}$

smooth $\mathbb{R}$ -algebras

Definition. Smooth $\mathbb{R}$ -algebras

A complete, geometric $\mathbb{R}$ -algebra A is called **smooth** if there exists a countable open covering $\{U_k\}$ of $|A|$ such that

$$A\Big|_{U_k} \cong \mathcal{C}^\infty(\mathbb{R}^n)$$

Gregory Moore - Differential Cohomology, TQFTs and beyond

[Home page of Greg Moore \(rutgers.edu\)](https://www.physics.rutgers.edu/~gmoore/)

Generalized Abelian Gauge Theories, Self-Duality, and Differential Cohomology, 2011

1. Minicourse of three lectures on Generalized Abelian Gauge Theories, Self-Duality, and Differential Cohomology, at the Simons Center Workshop on Differential Cohomology, Simons Center for Geometry and Physics, Stonybrook, Jan. 11-14, 2011. Here are [Lecture Notes](#) . Video available [here](#) .

✓ <https://www.physics.rutgers.edu/~gmoore/SCGP-Minicourse.pdf> ✓

```
frame: PDF
style: height: 900px;
urlSuffix: www.physics.rutgers.edu/~gmoore/SCGP-Minicourse.pdf
```

TASI School, 2023

1. On June 12,14,15,16, 2023 I gave four lectures at the [TASI School Aspects Of Symmetry](#)
 1. Handwritten lecture notes are available for [Lecture one on topological field theory](#) and [Lecture three on Generalized Abelian Gauge Theory](#) (really this was lecture 2), and
 2. Lecture 4 on [Differential cohomology](#) (really this was lectures 3 and 4). In addition there are notes for material I didn't get to: [on extended topological field theory and defects](#).

✓ <https://www.physics.rutgers.edu/~gmoore/TASI%20Lecture%201%20Introduction%20To%20Topological%20Field%20Theory-Final.pdf>

```
frame: PDF
style: height: 900px;
urlSuffix:
www.physics.rutgers.edu/~gmoore/TASI%20Lecture%201%20Introduction%20To%20Topological%20Field%20Theory-Final.pdf
```

✓ <https://www.physics.rutgers.edu/~gmoore/TASI%20Lecture%202%20Extensions%20And%20Applications%20Of%20TFT-Incomplete.pdf>

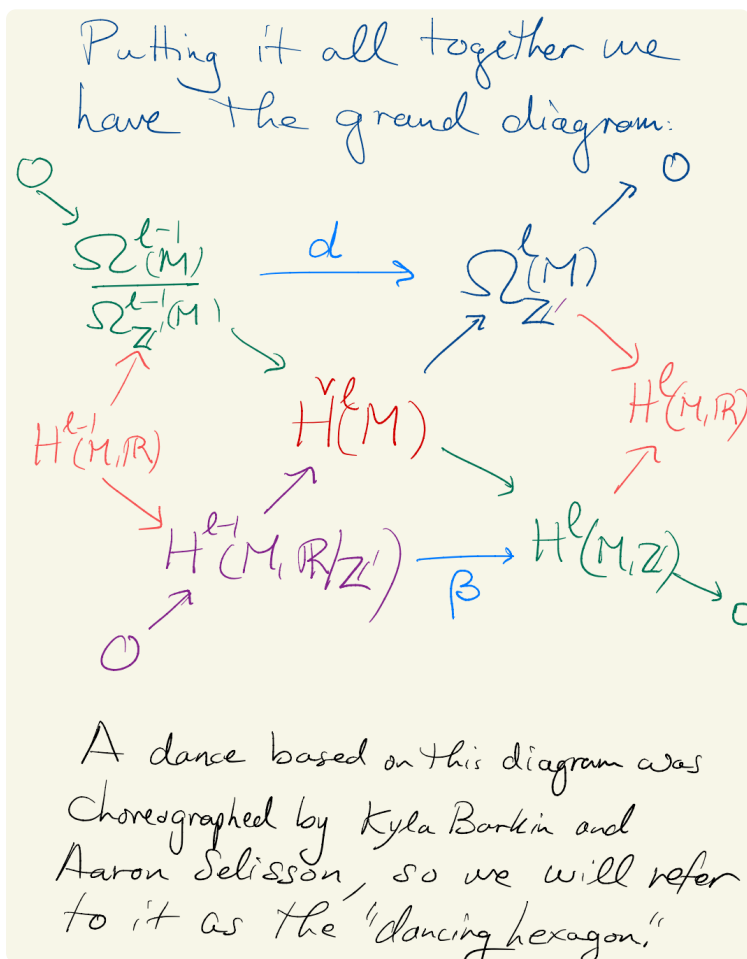
```
frame: PDF
style: height: 900px;
urlSuffix:
www.physics.rutgers.edu/~gmoore/TASI%20Lecture%202%20Extensions%20And%20Applications%20Of%20TFT-Incomplete.pdf
```

✓ <https://www.physics.rutgers.edu/~gmoore/TASI%20Lecture%204%20Differential%20Cohomology-Final.pdf> ✓

frame: PDF

style: height: 900px;

urlSuffix: www.physics.rutgers.edu/~gmoore/TASI%20Lecture%204%20Differential%20Cohomology-Final.pdf



✓ [TASI-GM-Reading.pdf](#) - Google Drive

frame: PDF

style: height: 900px;

urlSuffix: drive.google.com/file/d/1Ar8_2BCFtimZm5an93sU_yR38GCnBx0Q/preview

A Few Remarks On Topological Field Theory, 2015

✓ <https://www.physics.rutgers.edu/~gmoore/695Fall2015/TopologicalFieldTheory.pdf>

frame: PDF

style: height: 900px;

urlSuffix: www.physics.rutgers.edu/~gmoore/695Fall2015/TopologicalFieldTheory.pdf

notes

- [sett.Man.R.homology sing smooth.charac](#)

references

Andrew Neitzke - Lectures on BPS states and spectral networks

<https://gauss.math.yale.edu/~an592/exports/bps-lectures.pdf>

Given a Riemannian manifold

$$(M, g)$$

we have

(classical)	(quantum)
On T^*M the Hamiltonian flow of $H = \frac{\ p\ _g^2}{2} + V$	$L^2(M)$ the flow of the operator $H := \frac{1}{2}\Delta_g + V$

 Definition. $\mathcal{N} = 2, \dim = 1$ **SUSY Lie superalgebra**

We define a Lie superalgebra ($\mathbb{Z}_2$ -graded Lie algebra)

$$\mathcal{A} := \mathcal{A}^0 \oplus \mathcal{A}^1$$

as

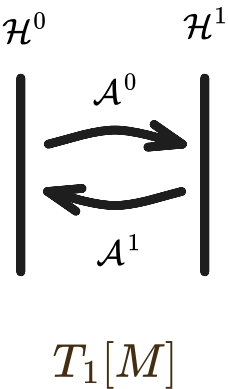
$$\begin{aligned} \mathcal{A}^0 &= \mathbb{C}H \\ \mathcal{A}^1 &= \mathbb{C}Q \oplus \mathbb{C}\bar{Q} \end{aligned}$$

with the brackets

$$\begin{aligned} [Q, \bar{Q}] &= 2H \\ [Q, H] &= 0 = [\bar{Q}, H] \end{aligned}$$

A $\mathbb{Z}_2$ -graded representation of $\mathcal{A}$ is a representation of $\mathcal{A}$ on a $\mathbb{Z}_2$ -graded vector space $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ where

$$\mathcal{A}^i : \mathcal{H}^j \rightarrow \mathcal{H}^{j+i}$$



 Definition.

$$T_1[M]$$

Consider the Laplace operator on differential forms

$$\Delta : \Omega_{L^2}^{\star}(M) \rightarrow \Omega_{L^2}^{\star}(M)$$
$$\Delta = [\mathrm{d}, \mathrm{d}^{\star}]$$

We have the following representation

$$\mathcal{H}^0 = \bigoplus_{k \geq 0} \Omega_{L^2}^{2k}(M)$$
$$\mathcal{H}^1 = \bigoplus_{k \geq 0} \Omega_{L^2}^{2k+1}(M)$$

and

$$Q \mapsto \mathrm{d}$$
$$\bar{Q} \mapsto \mathrm{d}^{\star}$$
$$H \mapsto \frac{1}{2} \Delta$$

- This representation is *unitary*.

•

$$2 \left\langle \psi, H\psi \right\rangle = \left\langle \psi, Q\bar{Q}\psi \right\rangle + \left\langle \psi, \bar{Q}Q\psi \right\rangle$$
$$= \|Q\psi\|^2 + \|\bar{Q}\psi\|^2$$

- Thus eigenvalues of H are non-negative.
- Moreover,

$$H\psi = 0 \iff Q\psi = 0 \text{ and } \bar{Q}\psi = 0$$

- In particular each

$$\psi \in \ker H = \ker Q \cap \ker \bar{Q}$$

generate

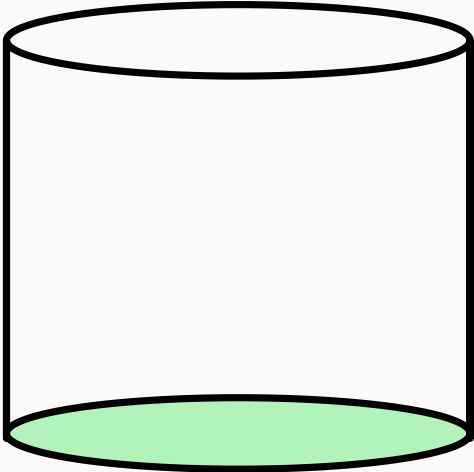
$$\mathbb{C}\{\psi\}$$

a 1-dim (trivial) representation of $\mathcal{A}$.

more examples

$\mathcal{N}$	dim	given		(co)homology
2	1	(M, g) is Riemannian	(QM) $T_1[M, g]$	de Rham
(2,2)	2	M is Kahler	(QFT) $T_2[M]$	Lagrangian Floer homology
(2,0)	6	$\mathfrak{g}$ be a ADE type Lie algebra, L be a link in $\mathbb{R}^3$	(CFT) $T_6[\mathfrak{g}, L]$	Kovanov homology

Sergei Gukov - Homological algebra of knots and BPS states

(co)homology	physics	index
	QM?	
de Rham $\cong$ simplicial	SUSY QM	Euler characteristic χ
quantum de Rham	2d QFT: A-model on $\mathbb{R} \times S^1$ 	Hirzebruch polynomial
Dolbeault $H_{\bar{\partial}}^{p,q}(X)$	B-model	genus?
Kovanov homology $\text{Kh}^{p,r}$	???	Jones polynomial $J(q)$
symplectic Floer homology $\text{HF}_{\text{sym}}(L_1, L_2)$	A-model on $\mathbb{R} \times I$ with boundary conditions L_1, L_2 (choice of Lagrangian submanifolds)	$ L_1 \cap L_2 $
instanton Floer homology $\text{HF}_{\text{sym}}({}^3M)$	4d gauge theory	Casson invariant

A Guest - From Quantum Cohomology to Integrable Systems

Let

$$\langle \ , \ \rangle : H^i(M; \mathbb{Z}) \times H_i(M; \mathbb{Z}) \rightarrow \mathbb{Z}$$

denote the natural pairing, and also the extended pairing

$$\langle \ , \ \rangle : H^*(M; \mathbb{Z}) \times H_*(M; \mathbb{Z}) \rightarrow \mathbb{Z}$$

where $\langle a, B \rangle = 0$ when $|a| \neq |B|$. In de Rham notation, $\langle a, B \rangle = \int_B a$. Since there is no torsion, these pairings are non-degenerate.

The *intersection pairing* is defined by

$$(\ , \) : H^*(M; \mathbb{Z}) \times H^*(M; \mathbb{Z}) \rightarrow \mathbb{Z}, \quad (a, b) = \langle ab, M \rangle = \int_M a \wedge b.$$

We have $\langle ab, M \rangle = \langle a, B \rangle = \langle b, A \rangle$. It follows that the intersection pairing $(\ , \)$ is a non-degenerate symmetric bilinear form.

3-point Gromov-Witten invariants

$$\langle A|B|C \rangle_0 = \langle abc, M \rangle = \int_M a \wedge b \wedge c =$$

Definition. Definition

Let M be a $\mathbb{C}$ -manifold and $p \in \mathbb{C}P^1$. Then

$$\mathcal{O}_D((\mathbb{C}P^1, p), (M, A))$$

Number fields

? Question

Is *algebraic number theory* study of *algebraic numbers* or subfield of number theory that uses *algebraic* tools?

algebraic numbers and integers

Definition. Algebraic numbers and integers

A **number field** is a finite extension

$$K \geq \mathbb{Q}$$

whose elements are **algebraic numbers** in K and an **algebraic integer** in K is a root $\alpha \in p^{-1}(0) \subset K$ of a monic polynomial $p \in \mathbb{Z}[X] \subset \mathbb{Q}[X] \subset K[X]$.

The set of algebraic integers in K is $\mathcal{O}_K$ called **ring of integers** of K .

[1]

Proposition: The minimal polynomial m_α of an algebraic number α has integer coefficients iff it is an algebraic number, that is,

$$m_\alpha(X) \in \mathbb{Z}[X] \iff \alpha \in \mathcal{O}_K$$

☀ $\implies$ is easy. So we assume α is an algebraic integer, so there is a monic polynomial $f \in \mathbb{Z}[X]$ such that

$$f(\alpha) = 0$$

and we know

$$\frac{f}{m_\alpha}(X) = g(X) \in \mathbb{Q}[X]$$

- Assume $g(X) \notin \mathbb{Z}[X]$ then there is a coefficient which in reduced form has a denominator $\neq 1 \implies$ there is a prime p which divides it.
- Let $u > 0$ be the largest integer such that $p^u g$ does not have any denominators divisible by p . And say $v > 0$ be the smallest integer such that $p^v h$ has no denominator divisible by p . We have

$$p^u g p^v h = p^{u+v} f$$

- Going modulo p we have

$$(p^u g \bmod p)(p^v h \bmod p) = 0 \in \mathbb{Z}_p[X]$$

where the left side is a product of non-zero polynomials $p^u g \bmod p$ and $p^v h \bmod p$ (non-zero as they do not have any denominators divisible by p), but it's product is zero, which contradicts $\mathbb{Z}_p$ being a field.

Corollary:

$$\mathcal{O}_{\mathbb{Q}} = \mathbb{Z}$$

☀ The minimal polynomial of $\frac{a}{b}$ is $X - \frac{a}{b}$ which has integer coefficients $\iff \frac{a}{b} \in \mathbb{Q}$.

☰ Let K be a number field and $\alpha \in K$. Then α is an algebraic integer $\iff$ the Abelian group

$$\mathbb{Z}[\alpha]$$

is finitely generated.

Corollary: Let K be a number field then $\mathcal{O}_K$ is a ring and

$$\mathbb{Q}\mathcal{O}_K = K$$

 **Example**

$$\mathbb{Z}\left[\frac{1}{2}\right]$$

is **not** finitely generated as $\frac{1}{2}$ is not an algebraic integer in $\mathbb{Q}$.

norms and traces

 **Definition. Norm and trace for a finite extension of number fields**

Let $L \geq K$ be a finite extension of number field K . Then we have an embedding

$$K \hookrightarrow \text{EndVec}_K(L)$$

as scalars. We call

$$N_{L/K}(\alpha) := \det(\alpha) \in K$$

as the **norm** of α and

$$\text{tr}_{L/K}(\alpha) := \text{tr}(\alpha)$$

as the **trace**.

- Norm is multiplicative and trace is additive as $\det, \text{tr}$ are like that resp.
- For $a \in K$ we have

$$N_{L/K}(a\alpha) = a^{\deg L/K} N_{L/K}(\alpha)$$

and

$$\text{tr}_{L/K}(a\alpha) = \text{tr}_{L/K}(\alpha)$$

Proposition: Let $K \leq L$ be an extension of number fields of degree n . Then there are n distinct embeddings

$$L \hookrightarrow \bar{K}$$

 **Definition. Conjugates**

Let $K \leq L$ be an extension of number fields of degree n . From

Proposition: Let $K \leq L$ be an extension of number fields of degree n . Then there are n distinct embeddings

$$L \hookrightarrow \bar{K}$$

we have

$$\sigma_1, \dots, \sigma_n$$

are n embeddings $L \hookrightarrow \bar{K} \subset \mathbb{C}$. Then for $\alpha \in L$ we call

$$\sigma_1(\alpha), \dots, \sigma_n(\alpha)$$

conjugates of α .

Proposition:

Definition. Conjugates

Let $K \leq L$ be an extension of number fields of degree n . From

Proposition: Let $K \leq L$ be an extension of number fields of degree n . Then there are n distinct embeddings

$$L \hookrightarrow \bar{K}$$

we have

$$\sigma_1, \dots, \sigma_n$$

are n embeddings $L \hookrightarrow \bar{K} \subset \mathbb{C}$. Then for $\alpha \in L$ we call

$$\sigma_1(\alpha), \dots, \sigma_n(\alpha)$$

conjugates of α .

Then for $\alpha \in L$ we have

$$N_{L/K}(\alpha) = \prod_{i=1}^n \sigma_i(\alpha)$$

$$\text{tr}_{L/K}(\alpha) = \sum_{i=1}^n \sigma_i(\alpha)$$

Corollary from norm and trace: Let K be a number field.

- $\alpha \in \mathcal{O}_K \implies$ norm and trace of α belongs to $\mathbb{Z}$, so

$$N, \text{tr} : \mathcal{O}_K \rightarrow \mathbb{Z}$$

- norm of α is ± 1 (unit in $\mathbb{Z}$) $\iff \alpha$ is a unit in $\mathcal{O}_K$

Proposition: Let K be a number field. Then

$$\mathcal{O}_K$$

is a free Abelian group of rank $[K, \mathbb{Q}]$.

Proposition: Let K be a number field. Then $\mathcal{O}_K$ is **integrally closed**.

ideal numbers

Definition. Let $I \subseteq R$ be a non-zero ideal, then the **norm** of I is

$$N(I) := \left| \frac{R}{I} \right|$$

Proposition: Let I be a non-zero ideal of $\mathcal{O}_K$ then $N(I)$ is finite and

$$N(\alpha \mathcal{O}_K) = |N_{K/\mathbb{Q}}(\alpha)|$$

 The set of all non-zero fractional ideals

$$I_K$$

of a number field K forms a group under multiplication.

discriminant

Definition. Signature of number fields

Let K be a number field of degree n with r_1 number of real embeddings and r_2 number of complex embeddings from

Proposition: Let $K \leq L$ be an extension of number fields of degree n . Then there are n distinct embeddings

$$L \hookrightarrow \bar{K}$$

Then

$$\text{sig}(K) := (r_1, r_2)$$

where $n = r_1 + 2r_2$.

Given n embeddings

$$\sigma_k : K \rightarrow \mathbb{C}$$

we have the product

$$\begin{aligned} \sigma : K &\rightarrow \mathbb{C}^n \\ x &\mapsto (\sigma_1(x), \dots, \sigma_n(x)) \end{aligned}$$

And given a $\mathbb{Z}$ -basis $\{\alpha_1, \dots, \alpha_n\}$ of $\mathcal{O}_K$ we have $n \times n$ matrix

$$[\sigma]_{\{\alpha_1, \dots, \alpha_n\}} = (\sigma_i(\alpha_j))_{i,j}$$

The determinant

$$\det(\sigma_i(\alpha_j))$$

computes "volume" of $K/\mathcal{O}_K$, which serves as a measure of *density* of $\mathcal{O}_K$ in K .

To get a real number we rather consider

$$\det([\sigma]_{\{\sigma_1, \dots, \sigma_n\}}^2) = \det \text{tr}_{K/\mathbb{Q}}(\alpha_i \alpha_j) \in \mathbb{Z}$$

Definition. Discriminant

Let $\alpha_1, \dots, \alpha_n \in K$. Then

$$\text{disc}(\alpha_1, \dots, \alpha_n) := \det \text{tr}_{K/\mathbb{Q}}[\alpha_i \alpha_j]_{i,j} \in \mathbb{Z}$$

In particular, if $\alpha_1, \dots, \alpha_n$ is a $\mathbb{Z}$ -basis of $\mathcal{O}_K$ then its disc is called the **discriminant Δ_K of K** .

Proposition: The symmetric bilinear form

$$\begin{aligned} K \times K &\rightarrow \mathbb{Q} \\ (x, y) &\mapsto \text{tr}_{K/\mathbb{Q}}(xy) \end{aligned}$$

is non-degenerate. Thus $\Delta_K \neq 0$.

Hence, $\Delta_K \neq 0$.

prime decomposition and ramification

Proposition: Let $\mathfrak{p} \subset \mathcal{O}_K$ be prime ideal. Then

- $N(\mathfrak{p}) \in \mathfrak{p} \cap \mathbb{Z}$
- $\mathfrak{p} \cap \mathbb{Z}$ is a **prime ideal** of $\mathbb{Z}$
- Hence,

$$\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$$

for some prime $p \in \mathbb{Z}$.

Definition. Prime ideals above primes

When

Proposition: Let $\mathfrak{p} \subset \mathcal{O}_K$ be prime ideal. Then

- $N(\mathfrak{p}) \in \mathfrak{p} \cap \mathbb{Z}$
- $\mathfrak{p} \cap \mathbb{Z}$ is a **prime ideal** of $\mathbb{Z}$
- Hence,

$$\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$$

for some prime $p \in \mathbb{Z}$.

we say **$\mathfrak{p}$ is above p**

```
\documentclass{standalone}
\usepackage{tikz-cd}
\usepackage{amsmath}
\usepackage{quiver}
\usepackage{mathrsfs}

\begin{document}
\tikzcdset{scale cd/.style={every label/.append style={scale=#1},cells={nodes={scale=#1}}}}
% \begin{tikzcd}[scale cd=0.8]

%
https://q.uiver.app/#q=WzAsMixbMCwwLCJcXG1hdGhmcmFre3B9XFxzdWJzZXQgXFxtYXRoc2Nye099X3tLfSBcXHN1YnNldCBLIl0sWzAsMSwicFxcbWF0aGJie1p9IFxcc3Vic2V0IFxcbWF0aGJie1p9IFxcc3Vic2V0IFxcbWF0aGJie1F9Il0sWzAsMSwiIiwuLHsic3R5bGUiOnsiaGVhZCI6eyJuYW1lIjoibm9uZSJ9fX1dXQ==
\begin{tikzcd}
    {\mathfrak{p}} \subset \mathscr{O}_K \subset K \\\
    {\mathfrak{p}} \subset \mathbb{Z} \subset \mathbb{Z} \subset \mathbb{Q} \\
    \arrow[no head, from=1-1, to=2-1]
\end{tikzcd}

\end{document}
```

We have the morphisms

$$\mathbb{Z} \xrightarrow{\iota} \mathcal{O}_K \rightarrow \mathcal{O}_K/\mathfrak{p}$$

whose kernel is

$$\{a \in \mathbb{Z} \mid a \in \mathfrak{p}\} = \mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$$

so by quotienting

$$\mathbb{Z}_p \hookrightarrow \mathcal{O}_K/\mathfrak{p}$$

which is a finite field extension of $\mathbb{Z}_p$.

🔑 **Definition.** The **inertial degree** of $\mathfrak{p} \subset \mathcal{O}_K$ **above** $p \in \mathbb{Z}$ is

$$f(\mathfrak{p}) := \dim_{\mathbb{Z}_p}(\mathcal{O}_K/\mathfrak{p})$$

$$N(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}| = p^{f(\mathfrak{p})}$$

🔑 **Definition.** **Ramification index** of $\mathcal{O}_K$

Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal **above** $p \in \mathbb{Z}$. We call **ramification index** of $\mathfrak{p}$, $e(\mathfrak{p})$, the power of $\mathfrak{p}$ that divides $p\mathcal{O}_K$. Hence,

$$p\mathcal{O}_K = \mathfrak{p}_1^{e(\mathfrak{p}_1)} \dots \mathfrak{p}_g^{e(\mathfrak{p}_g)}$$

We say p is **ramified** if $e(\mathfrak{p}_i) > 1$ for some i , otherwise p is non-ramified (that is, when $p\mathcal{O}_K = \mathfrak{p}_1 \dots \mathfrak{p}_g$).

Proposition: In

🔑 **Definition.** **Ramification index** of $\mathcal{O}_K$

Let $\mathfrak{p} \subset \mathcal{O}_K$ be a prime ideal **above** $p \in \mathbb{Z}$. We call **ramification index** of $\mathfrak{p}$, $e(\mathfrak{p})$, the power of $\mathfrak{p}$ that divides $p\mathcal{O}_K$.

Hence,

$$p\mathcal{O}_K = \mathfrak{p}_1^{e(\mathfrak{p}_1)} \dots \mathfrak{p}_g^{e(\mathfrak{p}_g)}$$

We say p is **ramified** if $e(\mathfrak{p}_i) > 1$ for some i , otherwise p is non-ramified (that is, when $p\mathcal{O}_K = \mathfrak{p}_1 \dots \mathfrak{p}_g$).

we have

$$[K : \mathbb{Q}] = \sum_{i=1}^g e(\mathfrak{p}_i) f(\mathfrak{p}_i)$$

💡 As

$$p\mathcal{O}_K = \mathfrak{p}_1^{e(\mathfrak{p}_1)} \dots \mathfrak{p}_g^{e(\mathfrak{p}_g)}$$

$$p^{[K:\mathbb{Q}]} = N(p\mathcal{O}_K) = N(\mathfrak{p}_1^{e(\mathfrak{p}_1)} \dots \mathfrak{p}_g^{e(\mathfrak{p}_g)})$$

and since norm is multiplicative and

$$N(\mathfrak{p}) = |\mathcal{O}_K/\mathfrak{p}| = p^{f(\mathfrak{p})}$$

we have

$$N(\mathfrak{p}_1^{e(\mathfrak{p}_1)} \dots) = \prod_i p^{f(\mathfrak{p}_i) e(\mathfrak{p}_i)}$$

Proposition: Let K be a number field, $\exists \theta$ such that

$$\mathcal{O}_K = \mathbb{Z}[\theta]$$

and $p \in \mathbb{Z}$ prime. Let

$$m_{\theta}(X) \bmod p = \prod_i \phi_i(X)^{e_i} \text{ in } \mathbb{Z}_p[X]$$

with ϕ_i coprime and irreducible. Then setting

$$\mathfrak{p}_i := (p, \tilde{\phi}_i(\theta)) = p\mathcal{O}_K + \tilde{\phi}_i(\theta)\mathcal{O}_K$$

where $\tilde{\phi}_i$ is any lift of ϕ_i to $\mathbb{Z}[X]$, gives us the factorization of $p\mathcal{O}_K = \mathcal{O}_K$

$$p\mathcal{O}_K = \mathfrak{p}_1^{e_1} \dots \mathfrak{p}_g^{e_g}$$

 Definition. Inert, totally ramified primes

	e	f	g	
inert	1	n	1	
totally ramified	n	1	n	

 Let K is a number field. Then p is **ramified** $\iff p \mid \Delta_K$.

[2]

Corollary of

 Let K is a number field. Then p is **ramified** $\iff p \mid \Delta_K$.

There are only a finite number of ramified primes for a number field K .

[3]

ideal class group

glossary

K	algebraic integers	elements	norm	trace	number of embeddings into $\mathbb{C}$	Δ_K
$\mathbb{Q}$	$\mathbb{Z}$					
$\mathbb{Q}(\sqrt{2})$	$\mathbb{Z}[\sqrt{2}]?$	$a + b\sqrt{2}$	$a^2 - 2b^2$	$2a$	2; $\sqrt{2} \mapsto \sqrt{2}, -\sqrt{2}$	
$\mathbb{Q}(i)$ that is $\mathbb{Q}(\sqrt{-1})$	$\mathbb{Z}?$					
$\mathbb{Q}(\sqrt{5})$	$\mathbb{Z}\left[\frac{1 + \sqrt{5}}{2}\right]$				2; $\sqrt{5} \mapsto \sqrt{5}, -\sqrt{5}$	5
$\mathbb{Q}(\sqrt{m})$	?					

number fields vs elliptic curves

number fields

$$K$$

elliptic curves

$$E := V(y^2 - f(x))$$

where $f(X)$ is a smooth cubic

(Analytic class number formula ~1839)

$$\lim_{s \rightarrow 0} \frac{\zeta_K(s)}{s^{r_1+r_2-1}} = -\frac{R_K h_K}{|\mu(K)|}$$

 **Birch and Swinnerton-Dyer conjecture**

Let $E(F)$ where F be a number field. Then analytic rank(E) = order at $s = 1$ of $L(E, s)$ = algebraic rank(E) = r .

(implicit in this conjecture is $L(E, s)$ is holomorphic at $s = 1$)

Finer version using other refined invariants of E

ramification in number theory and algebraic geometry

Proposition: Let D be any Dedekind domain with fraction field K , L/K a finite separable field extension and R the integral closure of D in L . Then R is again a Dedekind domain.

So

 **Definition. Ramified primes in Dedekind domains**

From

Proposition: Let D be any Dedekind domain with fraction field K , L/K a finite separable field extension and R the integral closure of D in L . Then R is again a Dedekind domain.

given any prime ideal $\mathfrak{p} \subset D$ of a Dedekind domain D , we have a primary decomposition

$$\mathfrak{p}R = \beta_1^{e_1} \dots \beta_r^{e_r}$$

where e_k is called the **ramification index** of the prime β_k . A prime $\mathfrak{p}$ is called **ramified** if some ramification index $e_k > 1$.

We say L/K is *unramified* if all prime ideals of R are *unramified* in L .

 **Definition. Ramification points of morphisms of curves**

Let X, Y be two curves (complete, non-singular, 1 dimensional integral scheme over $k = \bar{k}$). Let

$$f : X \rightarrow Y$$

be a finite morphism, $P \in X$ and $Q = f(P)$. Then f is **unramified** at P if

- the induced map on stalks

$$\mathcal{O}_{Y,Q} \rightarrow \mathcal{O}_{X,P}$$

gives

$$\mathfrak{m}_Q \mathcal{O}_{X,P} = \mathfrak{m}_P$$

- and the extension of residue fields

$$k(Q) \rightarrow k(P)$$

is separable

and otherwise it is **ramified** at P .

The map f is *unramified* if it is unramified at every point, and *ramified* if it is ramified at some point.

From

Proposition: Let D be any Dedekind domain with fraction field K , L/K a finite separable field extension and R the integral closure of D in L . Then R is again a Dedekind domain.

let

$$\iota : D \hookrightarrow R$$

induce the morphism

$$\iota^* : \text{Spec}(R) \rightarrow \text{Spec}(D)$$

Then a point $\mathfrak{p}$ is a ramification point of the map $\iff$ the prime $\mathfrak{p}$ is a ramified prime.

☀ Let $\mathfrak{p} \subset R$ be prime, that is $\mathfrak{p} \in \text{Spec}(R)$. Then

$$\mathfrak{p} \mapsto \mathfrak{q} \iff f(\mathfrak{p}) = \mathfrak{p} \cap D = \mathfrak{q}$$

[4]

1. <https://feog.github.io/antchap1.pdf> ↩
2. <https://math.stackexchange.com/a/700951> ↩
3. <https://feog.github.io/ANT10.pdf> ↩
4. https://www.wstein.org/edu/2010/581b/projects/matthew_ward/581bfinalproject.pdf ↩

Calabi-Yau manifolds

Definition. Calabi-Yau manifolds

A **Calabi-Yau manifold** is a compact Kahler manifold

$$(M, J, g)$$

such that either one of the following equivalent conditions occur

- $\text{Ric}(g) = 0$
- $c_1(M) = 0$
- $\text{Hol}(g) \subseteq SU(n)$
- the canonical bundle K_M is trivial
- there is a global, non-vanishing $(n, 0)$ -form

$\dim_{\mathbb{C}}$	name
1	Calabi-Yau elliptic curve
2	K3 surface
3	Calabi-Yau 3-fold

Hodge diamond

Proposition: The Hodge diamond of a Calabi-Yau 3-fold looks like

$$\begin{array}{ccccc} & & 1 & & \\ & & 0 & & 0 \\ & 0 & h^{1,1} & & 0 \\ 1 & h^{2,1} & & h^{2,1} & 1 \\ & 0 & h^{1,1} & & 0 \\ & & 0 & & 0 \\ & & 1 & & \end{array}$$

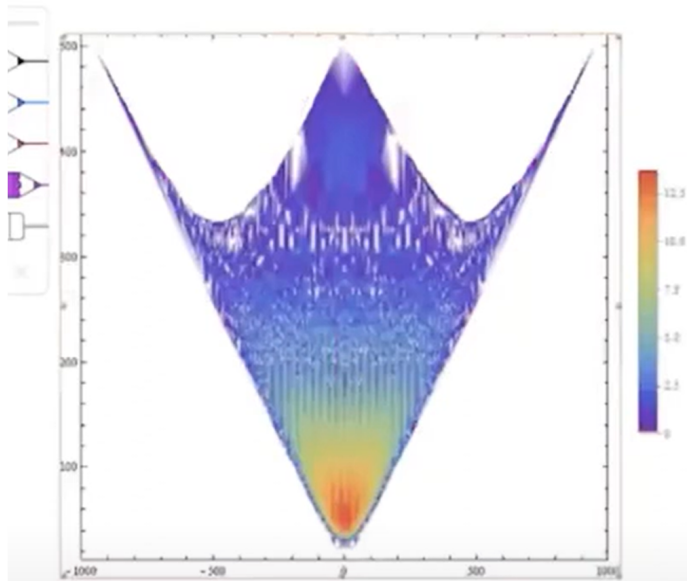
Mark - what is mirror symmetry

[What is mirror symmetry? \(youtube.com\)](https://www.youtube.com/watch?v=...)

(version 1.0)

Mirror Symmetry Conjecture: For each (a, b) appearing as the Hodge numbers of some CY3, M , there is a different CY3, $\tilde{M}$ with Hodge numbers (b, a) .

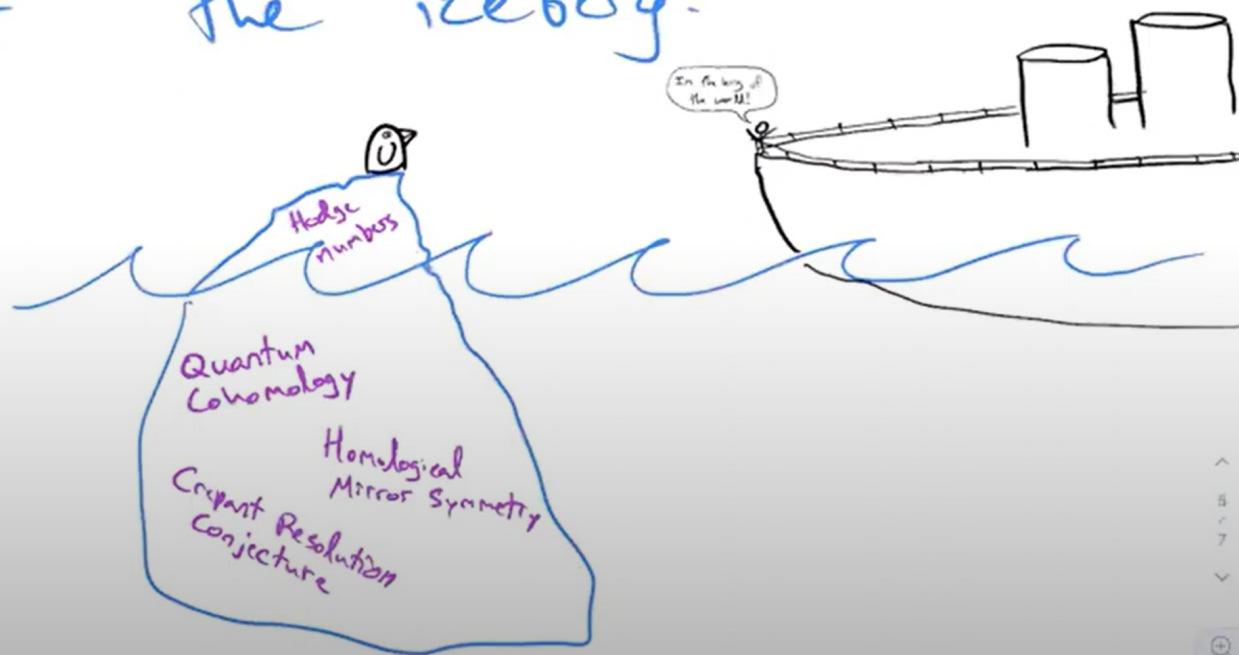
✓
mirror manifold



- heat map of $473, 800, 652$
- the symmetry seems to persist.
- what's with those parabolas??

current database of CY3s: $\sim 10^{10}$ distinct examples

It turns out this numerical symmetry is just the tip of the iceberg.



(versions 1, 2, 3, 4...)

Mirror Symmetry: Every CY3, M , has a mirror partner, $\check{M}$, such that:

M	$\check{M}$
① $a = h^{1,1}(M)$	$b = h^{2,1}(\check{M})$
② symplectic structure	complex structure
③ # of curves in M (★)	period integrals of $\check{M}$
④ Lagrangian submanifolds in M	vector bundles on $\check{M}$
⋮	⋮

(★) this is algebraic geometry

- Q's:
1. Given M , how to find $\check{M}$? (toric geometry)
 2. What about other dimensions?
 3. non-Calabi-Yau?
 4. How does changing the geometry of M affect the quantum invariants above?

- Q's:
1. Given M , how to find $\check{M}$? (toric geometry)
 2. What about other dimensions?
 3. non-Calabi-Yau?
 4. How does changing the geometry of M affect the quantum invariants above?
- The same symmetry occurs if we expand our notion of what a geometric space is.

logbook

• 6 May

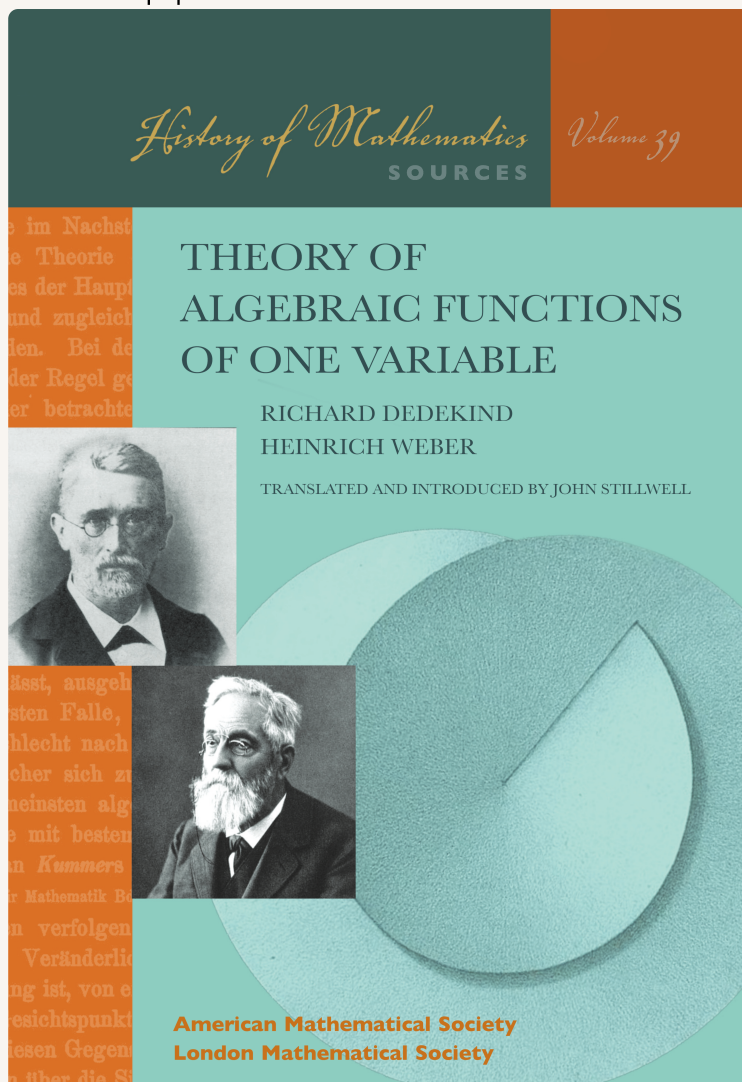
#logbook/spring25

- Removed fluid dynamics from the notebook.
- Changed the pdf "hosting" site, now it shows bookmarks on the left!
- Added a page for [Young Mathematicians' Symposium 2025, IISER Mohali](#)
- Added [Uri Bader - A course in linear groups and ergodic theory](#)

• 14 April

#logbook/spring25

- Was studying branched covering properties of $z^3 - 3z$
 - [Cubic function fields with prescribed ramification](#)
 - [Mating quadratic maps with the modular group II](#)
- Adding more topics in [complex 1-manifolds, AKA Riemann surfaces](#)
- Looking into [Dedekind-Weber theory](#)
 - <https://mathoverflow.net/a/176887>
 - [The History of Algebraic Geometry](#)
 - the classic paper



- [tifr18.pdf](#)
- [Mathematics_1951_29809_Introduction to the theory of algebraic functions of one variable.pdf](#)

- [riemann-surfaces/riemann_surfaces.pdf at master · svenpruefer/riemann-surfaces](#)
- [Posts matching 'polynomial branched covering' - Mathematics Stack Exchange](#)
 - [algebraic curves - Computing the monodromy for a cover of the Riemann sphere \(and Puiseux expansions\) - Mathematics Stack Exchange](#)
 - [algebraic curves - Computing the monodromy for a cover of the Riemann sphere \(and Puiseux expansions\) - Mathematics Stack Exchange](#)
 - [complex analysis - For which polynomials \$p \in \mathbb{C}\[w\]\$ are the branches of the inverse \$p^{-1}\$ expressible using algebraic operations? - Mathematics Stack Exchange](#)
- [qf08sol.pdf](#)
- [RS2L25.pdf](#)

- 1 April

references

smooth manifolds

#reference smooth manifolds, *continued*

[What is a Manifold? - Mikhail Gromov - YouTube](#)

- main ref
 - [John M Lee.Introduction to smooth manifolds-Springer \(2013\).pdf](#)
- DEs
 - [REF differential equations](#)
 -
- sheaves, algebra
 - [Torsten Wedhorn - Manifolds, Sheaves, and Cohomology-Springer \(2016\).pdf](#)
 - [Jet Nestruev - Smooth Manifolds and Observables-Springer \(2003\).pdf](#)
 - [DiffGeo_Waldmann.pdf](#)
 - [Liviu I. Nicolaescu - Lectures on the Geometry of Manifolds.pdf](#)
- algebraic topology / differential topology
 - [Milnor - Topology from the Differentiable Viewpoint \(1988\).pdf](#)
 - [Bott Tu - Differential forms in algebraic topology](#)
 - [Vinicius Ramos-Differential Topology](#)
 - [Vinicius Ramos-Topology of Manifolds](#)

• *Differential Topology*

 **A saying in MathOverflow**

Run don't walk your way to [Milnor's Topology from the Differentiable Viewpoint](#)

- [Differential topology - difftop.pdf \(ethz.ch\)](#)
 - follows Milnor and completes exposition
- Differential Topology by Guillemin and Pollack

- geometric topology

- differential forms
 - [Bott Tu - Differential forms in algebraic topology](#)
 - [Isu Vaisman – Differential Forms & Cohomology](#)
 - Morita differential forms

PDEs, geometric analysis, harmonic analysis

• PDEs

- [Doctorate course: Partial Differential Equations and Applications \(IMPA, 2022\)](#) with [notes in Portuguese](#)

• Ricci flow

- <https://www.mfo.de/about-the-institute/staff/prof-dr-gerhard-huisken/lectures/introduction-to-ricci-flow>
- https://www.dropbox.com/scl/fo/le0chdpiexj2i0yys6c/ADkJqSxPmNu3NGu9FOm_ayl/recordings?rlkey=rdva7s9rqdnrjqcmvsp33xih4&subfolder_nav_tracking=1&st=4rl2afuu&dl=0

REF spectral geometry

#reference spectral geometry

- [Olivier Lablée. - Spectral Theory in Riemannian Geometry \(2015, European Mathematical Society\).pdf](#)

Moreover, spectral geometry is an inter-disciplinary field of mathematics; it involves

- analysis of ODE and PDE
- dynamical systems: classical and quantum completely integrable systems, quantum chaos, geodesic flows and Anosov flows on (negatively curved) manifolds (for example, Ruelle resonances are related to the spectrum of the Laplacian, see the recent articles [\[Fa-Ts1\]](#), [\[Fa-Ts2\]](#)).
- geometry and topology (the main purpose of these notes is to explain this)
- geometric flow: the parabolic behaviour of scalar curvature

$$\frac{\partial R_{g(t)}}{\partial t} = \Delta R_{g(t)} + 2 |\text{Ric}_{g(t)}|^2;$$

the Ricci flow (see also Sections [7.8.3](#) and [7.8.4](#))

$$\frac{\partial g}{\partial t} = -2\text{Ric}_{g(t)}$$

- probability: the Brownian motion on a Riemannian manifold (M, g) is defined to be a diffusion on M (generator operator is given by $\frac{1}{2}\Delta_g$).
- etc. ...

- <https://arxiv.org/pdf/1301.5525>
- <https://arxiv.org/pdf/1311.4932>
- <https://www.michaellevitin.net/Book/TSG230529.pdf>
- <https://math.mit.edu/~dyatlov/files/2015/restalk.pdf>
- [Semyon Dyatlov's Homepage](#)

• fourier/harmonic analysis, geometric measure theory

- [Abstract harmonic analysis](#)
- [HIM Lectures: Trimester Program "Harmonic Analysis and Partial Differential Equations"](#)
- [Programa de Doutorado: Topics in Analysis: Sphere packings, Fourier analysis and beyond \(2018\)](#)
- [Lectures on Harmonic Analysis - YouTube](#)
- [IPS on Geometric Measure Theory, Harmonic Analysis, Partial Diff. Equations - MAA MathFest 2023](#)
- restriction problem
 - [Hong Wang: The restriction problem and the polynomial method, Lecture I](#)
 - [The Polynomial Method and the Restriction Problem - Larry Guth](#)
 - [Larry Guth \(MIT\) - 1/3 Introduction to decoupling \[MSRI 2017\]](#)
- [L01 Harmonic analysis and affine Hecke algebras by Eric Opdam](#)
- [Eugenia Malinnikova: Frequency function methods in PDE I](#)

geometry, groups and dynamics

"REF dynamics.Anosov flows" could not be found.

solitons

-
- [Solitons III \(2020-21\) Michaelmas term - YouTube](#)
- [bose einstein condensate - Link between integrability and soliton solutions - Physics Stack Exchange](#)
- https://www.maths.dur.ac.uk/users/stefano.cremonesi/pictures_animations_Solitons/Solitons_notes_2023_24.pdf

- [Advanced School & Workshop on Nonlocal Partial Differential Equations and Applications to Geometry, Physics and Probability | \(smr 3118\) - YouTube](#)

geometry

REF algebraic geometry

#reference

algebraic geometry

- https://www.math.stonybrook.edu/~kamenova/homepage_files/Hartshorne_engl.pdf
- <https://ncatlab.org/nlab/show/books+in+algebraic+geometry>
- Andreas Gathmann
 - [Andreas Gathmann - Class Notes: Algebraic Geometry \(rptu.de\)](#)
 - [Andreas Gathmann-Plane Algebraic Curves Andreas Gathmann - Class Notes: Plane Algebraic Curves \(rptu.de\)](#)
- [Joe Harris \(mathematician\) - Wikipedia](#), Griffiths
 - complex
 - [Phillip Griffiths, Joseph Harris, - Principles of algebraic geometry-Wiley \(2011\).pdf](#)
 - [Introduction to Algebraic Curves\(Griffins\).pdf](#)
 - [E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris - Geometry of Algebraic Curves_ Volume I \(1985\).pdf](#)

-  [1] The reader is assumed to have a working knowledge of basic algebraic geometry such as is given, for example, in the first chapter of Hartshorne's book Algebraic Geometry.

- [Enrico Arbarello, Maurizio Cornalba, Phillip A. Griffiths - Geometry of Algebraic Curves_ Volume II with a contribution by Joseph Daniel Harris \(2011\).pdf - moduli](#)
- [287y.pdf](#)

- algebraic

- Hartshorne Ch 1~ [Joe Harris - Algebraic Geometry_ A First Course \(1992, Springer\).pdf](#)
- [The Geometry Of Schemes Eisenbud Joe Harris 2000 \(2013\).pdf](#)
- [3264 and All That: A Second Course in Algebraic Geometry](#)
<https://scholar.harvard.edu/files/joeharris/files/000-final-3264.pdf>
- [Joe Harris & Ian Morrison - Moduli of Curves.pdf](#)
- Starts from Hartshorne Ch 4~ [David Eisenbud_ Joe Harris - The Practice of Algebraic Curves_ A Second Course in Algebraic Geometry \(2024, American Mathematical Society\).pdf](#)

- Fulton Alg Curves
- geometric invariant theory
- [Tamás Szamuely - Galois groups and fundamental groups-Cambridge University Press \(2009\).pdf](#)
- [Lectures on Étale Cohomology by J.S. Milne](#)
- [Sebastian Casalaina-Martin, Jonathan Wise - An introduction to moduli stacks, with a view towards Higgs bundles on algebraic curves \(2017\)](#)
- [note0405.ps - Google Drive](#)

[2]

commutative algebra

Richard E Borcherds.Commutative algebra and algebraic geometry

Richard E Borcherds: Commutative algebra and algebraic geometry

Commutative and homological algebra

- [sett.field.poly n.permutation action](#)

Algebraic geometry

- [Algebraic geometry I: Varieties](#)
- [Algebraic geometry II: Schemes](#)
- [Algebraic geometry: extra topics](#)

[Eisenbud, David - Commutative Algebra_ with a View Toward Algebraic Geometry-Springer New York \(2004\).pdf](#)

complex analysis

Quote

[Advanced Complex Analysis - Harvard](#)

Complex analysis is a nexus for many mathematical fields, including:

1. Algebra (theory of fields and equations);
2. Algebraic geometry and complex manifolds;
3. Geometry (Platonic solids; flat tori; hyperbolic manifolds of dimensions two and three);
4. Lie groups, discrete subgroups and homogeneous spaces (e.g. $H/SL_2(\mathbb{Z})$);
5. Dynamics (iterated rational maps);
6. Number theory and automorphic forms (elliptic functions, zeta functions);
7. Theory of Riemann surfaces (Teichmüller theory, curves and their Jacobians);
8. Several complex variables and complex manifolds;
9. Real analysis and PDE (harmonic functions, elliptic equations and distributions).

- [Ahlfors-complex-analysis-third-edition-1979.pdf](#)
- [Walter Rudin - Real and complex analysis-McGraw-Hill \(1987\).pdf](#)
- [Sarason-ComplexFunctionTheory.pdf](#)

[T. E. Venkata Balaji - An Introduction to Families, Deformations and Moduli](#)

differential equations in complex domain

- with monodromy [ordinary differential equations - Diff Eq Textbook with Monodromy? - Mathematics Stack Exchange](#)
 - E. Hille, [Ordinary differential equations in the complex domain](#). It has a section 5.7 (with exercises in the end) on monodromy. [Einar Hille - Ordinary Differential Equations in the complex domain-Dover Publications \(1997\).pdf](#)
 - For more challenging reading: Y. Ilyashenko, S. Yakovenko, [Lectures on Analytic Differential Equations](#). Chapter III deals with monodromy.
 - Or you can try D. Anosov, A. Bolibruch, [The Riemann-Hilbert Problem](#). The book works out many examples but **is not** a textbook.
 - Lastly: H. Zoladek, [The Monodromy Group](#). [Hendryk Zoladek - The monodromy group \(2006\).pdf](#)
 - Deligne's book.
- <https://www.math.utah.edu/~milicic/Eprints/de.pdf>
- https://archive.mpim-bonn.mpg.de/id/eprint/3941/1/preprint_2020_26.pdf#page=5.17
- <https://www.math.mcgill.ca/goren/Curves/Riemann-Hilbert.pdf>
- fields.utoronto.ca/programs/scientific/08-09/o-minimal/geometry/Yakovenko.pdf

complex geometry

[Why are the two kinds of algebraic geometry \(complex manifolds/schemes\) considered the same subject? : r/math \(reddit.com\)](#)

[cv.complex variables - Complex analytic vs algebraic geometry - MathOverflow](#)

- [Complex geometry - Wikipedia](#)

- complex

- [Phillip Griffiths, Joseph Harris, - Principles of algebraic geometry-Wiley \(2011\).pdf](#)
- [Introduction to Algebraic Curves\(Griffins\).pdf](#)
- [E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris - Geometry of Algebraic Curves__ Volume I \(1985\).pdf](#)

-  ^[1] **The reader is assumed to have a working knowledge of basic algebraic geometry such as is given, for example, in the first chapter of Hartshorne's book Algebraic Geometry.**

- [Enrico Arbarello, Maurizio Cornalba, Phillip A. Griffiths - Geometry of Algebraic Curves__ Volume II with a contribution by Joseph Daniel Harris \(2011\).pdf - moduli](#)
- [287y.pdf](#)

-
1. [E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris - Geometry of Algebraic Curves__ Volume I \(1985\).pdf > page=13&selection=86,0,109,1 <](#)

🔍 Literature recommendations in between Algebraic and Differential Geometry : r/math (reddit.com)

Read Griffiths & Harris *Principles of Algebraic Geometry* chapters 0,1,2. Especially chapter 2 is beautiful.

For complex geometry, read Huybrechts or Wells' *Differential Analysis on Complex Manifolds* (which is more analytically focused).

You won't be able to escape proper analysis for the **Hodge theorem on Kahler manifolds**: its simply unavoidable. Modern Hodge theory in algebraic geometry is a very advanced subject. You can study it without actually learning the proof of the Hodge theorem proper, but if you're going to do so you better be doing a PhD in algebraic geometry, in which case your supervisor will instruct you on where to learn. For actual proofs of the Hodge theorem, I learned it from Warner *Foundations...* but that's pretty dated. The "true" proof uses pseudodifferential operators and is mostly of interest to people looking to go into geometric analysis, and there are other cool proofs (such as in Roe's *Elliptic operators, topology, and asymptotic methods*) but none of them will give great algebraic insights. If you're not interested in spending 2 months learning elliptic operator theory in some depth then you're better off just reading what you see in Huybrechts or another book on Kahler manifolds (I love Ballmann *Lectures on Kahler Manifolds*) so you know all the consequences of the Hodge theorem.

Once you have a good foundation in complex geometry, it's much easier to see which direction you want to go in.

- On the more algebraic side people are doing stability conditions, enumerative geometry, toric mirror symmetry, etc. which involve plenty of complex-geometric thinking but are mainly algebraic.
- A fun thing to start reading might be *Lectures on Bridgeland Stability* by Macri to see if you like it.
- On the more analytic side, people are interested in using algebraic+differential techniques to solve differential equations (canonical metrics on varieties, vector bundles) ^[1] but usually the algebraic input is *fairly* limited (you sort of know it ahead of time, and then all the true difficulty is in analysis). The standard books here are Gabor's *Introduction to Extremal Kahler metrics* and Kobayashi *Differential geometry of complex vector bundles* to get started.

Keep in mind that to appreciate these modern threads of the theory there are a lot of things you need to "know" to really "get it," but not to get going. You'll want to eventually have an appreciation for:

- geometric invariant theory,
- symplectic reduction,
- construction of moduli spaces in both algebraic and differential geometry,
- the origins of enumerative invariants via integration over moduli spaces,
- some standard examples of moduli (Riemann surfaces,
- vector bundles over Riemann surfaces, starting with the Jacobian (see Chapter 2 of Griffiths&Harris) etc.

Due to its intersectional nature, complex geometry has a lot more of this background information to learn than most other subjects, but you don't have to learn it all at once. For example note Gabor's book doesn't get to the thing he's really interested in (K-stability) until chapter 6, because of all the background knowledge he has to go through.

Another very new area in the modern theory is non-Archimedean geometry. This is being used by pure arithmetic geometers by Scholze and others, but its also growing in popularity in traditional complex geometry due to the works of people like Boucksom et al (in the study of canonical Kahler metrics) and Yang Li (in the study of the SYZ conjecture). It seems to be one of the only new tools to really understand Calabi-Yau manifolds, which is the next unknown in this branch of complex geometry, so it's likely to be a hot topic for a little while. It has a very "pure algebraic geometry" flavour, as you're basically doing Hartshorne-style stuff over non-Archimedean fields. Somehow it produces fairly concrete results though.

If all of that seems overwhelming, then the correct thing to do is go read several of Donaldson's surveys of the subject, which will be both interesting and inspiring.

🔍 [Textbook recs for intro to complex algebraic geometry : r/math \(reddit.com\)](#)

- Griffiths & Harris *Principles of Algebraic Geometry* (There are better CG intros than Chapter 0, but chapters 1 and 2 are essential reading, and the later chapters are great for building examples to help you understand the theory) [Phillip Griffiths, Joseph Harris, - Principles of algebraic geometry-Wiley \(2011\).pdf](#)
- Voisin *Hodge theory and complex algebraic geometry I and II* (second volume is very hard, special focus on Hodge theory/Hodge structures and deformation theory)
- Huybrechts *Complex geometry* (good first book in CG)
- Eisenbud & Harris *3264 and all that* (good second book, a good introduction to intersection theory which is a super important tool in CAG)
- Demailly *Complex analytic and differential geometry* (good if you want to do geometric analysis on complex manifolds)
- Wells *Differential analysis of complex manifolds* (good first book)
- Székelyhidi *An introduction to extremal Kähler metrics* (excellent book, specifically about K-stability, includes a complete proof of the Calabi conjecture)
- Vakil *Rising sea* (this is more pure algebraic geometry)
- Huybrechts *Lectures on K3 surfaces* (good book, again good for seeing techniques and examples)
- Clay Mirror symmetry book from 2000 (bad book, but has a completely comprehensive state of play of mirror symmetry from 20 years ago; later chapters 39-40 are good reading even if you don't understand anything in detail) [clay-Mirror-Symmetry.pdf](#)
- Huybrechts & Lehn *Moduli spaces of sheaves* (good introduction to sheaf-theoretic things, lots of important ideas that you should be familiar with here)

There have been some big advancements in complex algebraic geometry over the last 25 years in different directions, and no one source covers all of it. Some big recent topics are: Classification of Fano threefolds, K-stability, Bridgeland stability conditions, derived complex algebraic geometry, birational geometry, Gromov--Witten theory, and mirror symmetry. There have been some results in pure Hodge theory but I don't know too much about this (have a look at some survey's, perhaps of Voisin, but she tends to write in French as much as English).

There is generally no good single reference for understanding mirror symmetry, which is a broad subject, and you are better off looking up various surveys and lecture notes. The book *Calabi-Yau manifolds and related geometries: lectures at a summer school in Nordfjordeid, Norway* is a possible starting point after you've studied complex manifolds a bit more deeply, and this considers the SYZ conjecture. You can also come at the topic from the pure algebraic geometry perspective through derived categories, which I know less about, or you can look into Gromov--Witten theory and toric mirror symmetry.

The subject requires a lot of moving in and out of various areas and reading the key papers, rather than reading a textbook or two and gaining a comprehensive understanding of what is going on. You can have a look at some survey's and lecture notes of people like Donaldson, Dominic Joyce, Richard Thomas from the last 20 years. If you have a specific area you have in mind I may be able to suggest some key papers.

If you want to get into the singular theory that involves a lot of study by example. I don't know good reference texts for this but you will want to look at works by birational geometers who have learned many techniques about how to deal with singular varieties. Key tools are blow ups, $\mathbb{Q}$ -divisors, intersection theory, normalisation, and cohomology calculations. You generally learn how to do these things by reading the references above and by

reading papers where people study many examples. If you want to see such things in action look up papers by Ivan Cheltsov to get a feel for it.

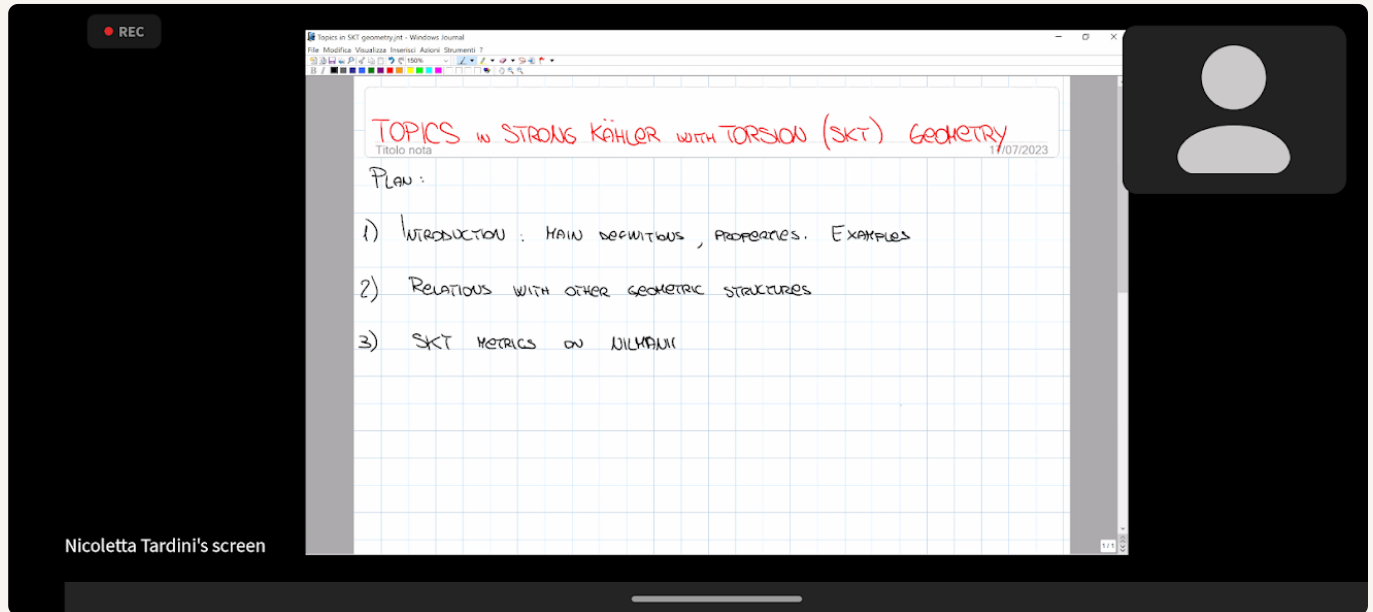
- [Intersection Theory \(1 of 5\) - YouTube](#) [Intersection theory in algebraic geometry - lccs \(columbia.edu\)](#)

Kahler manifolds

- <http://people.maths.ox.ac.uk/~joyce/Nairobi2019/BallmannKahlerManifolds.pdf>
- <https://hal.science/file/index/docid/1136/filename/kg.pdf>

SKT geometry

- [Nicoletta Tardini - Lectures on strong Kähler with torsion \(SKT\) metrics - YouTube](#)



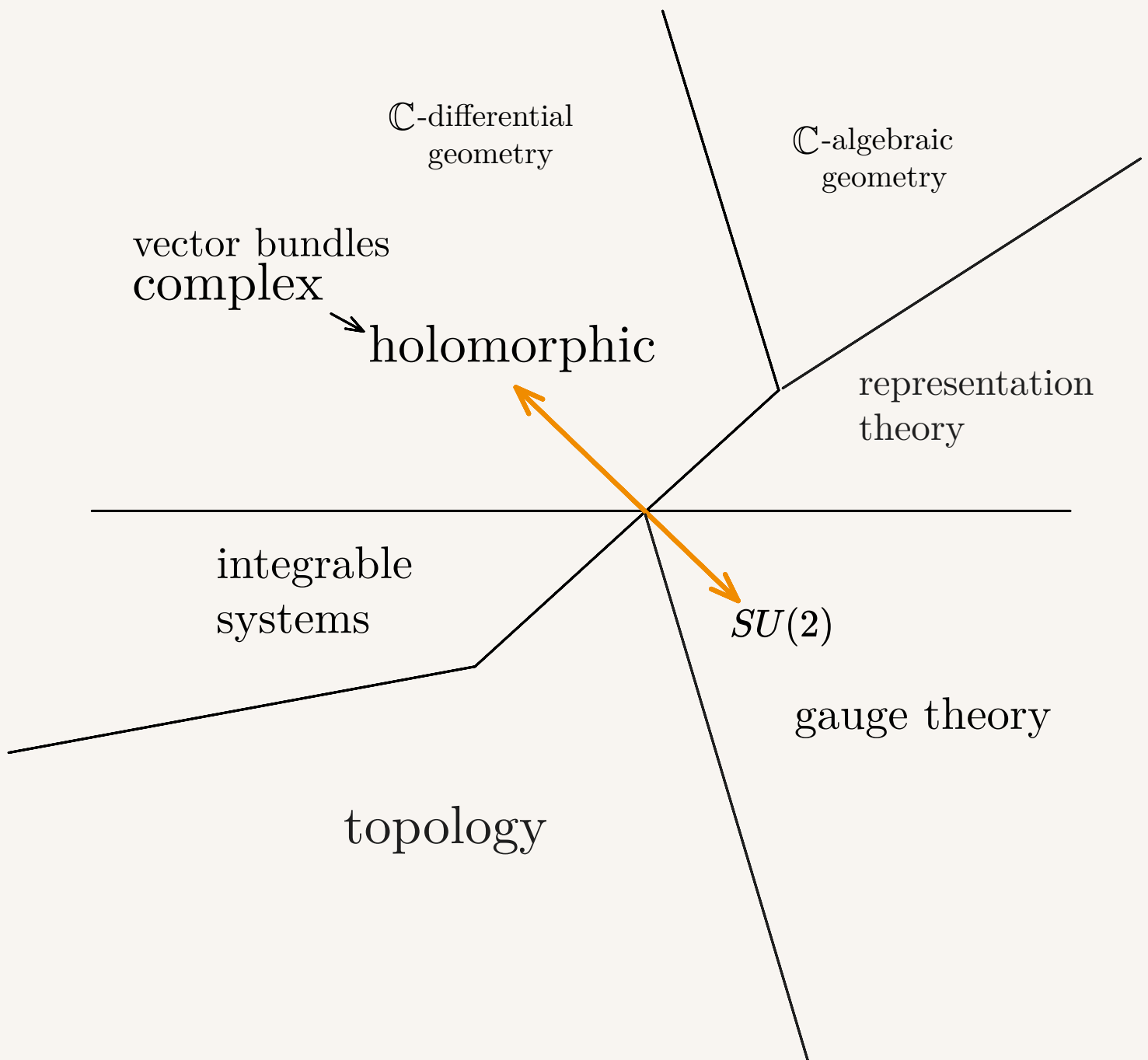
- [Hodge theory of SKT manifolds - ScienceDirect](#)

[SKT Geometry](#) | [PIRSA](#)

1. [REF cpx vec bun x gauge x geometry](#) ↩

🗨 [Literature recommendations in between Algebraic and Differential Geometry : r/math \(reddit.com\)](#)

On the more analytic side, people are interested in using algebraic+differential techniques to solve differential equations (canonical metrics on varieties, vector bundles) but usually the algebraic input is fairly limited (you sort of know it ahead of time, and then all the true difficulty is in analysis). The standard books here are Gabor's *Introduction to Extremal Kahler metrics* and Kobayashi *Differential geometry of complex vector bundles* to get started.



- $\mathbb{C}$ -algebraic geometry $\leftrightarrow$ $\mathbb{C}$ -differential geometry
- Vector bundles $\leftrightarrow$ gauge theories
 - [Kobayashi–Hitchin](#)
- "integrable geometry"

• 🗨 [How to get into integrable systems? : r/math \(reddit.com\)](#)

I've only had a tangential interaction with integrable systems. Specifically, I study what *you might call integrable geometry*. The "niceness" that makes a system integrable (spectral deformation, Backlund transformations, etc.) corresponds to nice transformations/deformations of a certain family of submanifolds (e.g. the sine-Gordon equation and K-surfaces). For me this really boils down to a there being family of flat connections somewhere so I haven't really ever had to get into the PDE side of things.

From this perspective, the most useful thing I ever read on integrable systems was the book **Integrable Systems by Hitchin, Segal and Ward**. If nothing else, read Hitchin's introduction.

books

- [Griffiths and Harris-Alg geom](#)
- [John M. Lee - Introduction to Complex Manifolds-American Mathematical Society \(2024\).pdf](#)

relating everything

- https://impa.br/wp-content/uploads/2017/04/25CBM_02.pdf
- <https://walpu.ski/Teaching/WS2122/DifferentialGeometry3/Topics.pdf>
- walpu.ski/Teaching/WS2324/DifferentialGeometry3/GaugeTheoryS.pdf

Integrable systems and complex geometry

- <https://arxiv.org/pdf/0706.1579>

Hitchin and Higgs

Kobayashi–Hitchin

[Kobayashi–Hitchin correspondence - Wikipedia](#)

In differential geometry, algebraic geometry, and gauge theory, the **Kobayashi–Hitchin correspondence** (or Donaldson–Uhlenbeck–Yau theorem) relates **stable vector bundles over a complex manifold** to **Einstein–Hermitian vector bundles**. The correspondence is named after Shoshichi Kobayashi and Nigel Hitchin, who independently conjectured in the 1980s that the moduli spaces of stable vector bundles and Einstein–Hermitian vector bundles over a complex manifold were essentially the same. This was proven by Simon Donaldson for projective algebraic surfaces and later for projective algebraic manifolds, by Karen Uhlenbeck and Shing-Tung Yau for compact Kähler manifolds, and independently by Buchdahl for non-Kähler compact surfaces, and by Jun Li and Yau for arbitrary compact complex manifolds. The theorem can be considered a vast generalisation of the Narasimhan–Seshadri theorem concerned with the case of compact Riemann surfaces, and has been influential in the development of differential geometry, algebraic geometry, and gauge theory since the 1980s. In particular the Hitchin–Kobayashi correspondence inspired conjectures leading to the nonabelian Hodge correspondence for Higgs bundles, as well as the Yau–Tian–Donaldson conjecture about the existence of Kähler–Einstein metrics on Fano varieties, and the Thomas–Yau conjecture about existence of special Lagrangians inside isotopy classes of Lagrangian submanifolds of a Calabi–Yau manifold.

Simon Donaldson

- <https://www.ma.ic.ac.uk/~skdona/EMP.PDF>

-
- Donaldson's recent work centers on a problem in complex differential geometry concerning a conjectural relationship between algebro-geometric "stability" conditions for smooth projective varieties and the existence of "extremal" Kähler metrics, typically those with constant scalar curvature (see for example cscK metric). ^[1]
 - Donaldson obtained results in the toric case of the problem (see for example Donaldson (2001)).
 - He then solved the Kähler–Einstein case of the problem in 2012, in collaboration with Chen and Sun. This latest spectacular achievement involved a number of difficult and technical papers.
 - The first of these was the paper of Donaldson & Sun (2014) on Gromov–Hausdorff limits.
 - The summary of the existence proof for Kähler–Einstein metrics appears in Chen, Donaldson & Sun (2014). Full details of the proofs appear in Chen, Donaldson, and Sun (2015a, 2015b, 2015c).

1. https://en.wikipedia.org/wiki/Simon_Donaldson#Research ↩

- [Simon Donaldson - Einstein-Kahler, algebraic geometry](#)

Siberg-Witten

Quote

Maybe Seiberg-Witten theory. In the 80s, topologists studied manifolds using the Donaldson equations. This was grueling work, but they learned wondrous things. In 1994, Witten suggested the Seiberg-Witten equations, which were much easier to study. /

1:26 PM · Aug 21, 2024

Practically overnight, all the major results of Donaldson theory were reproven with 1/10th the length, plus many more exciting things. The old tools were obsolete.

Nowadays, the donaldson equations are making a comeback, doing some things the Seiberg-Witten eqs can't. Not a permanent usurping. But for those in the room where it happened, topology turned on its head.

Here is an email from that time. It's electric.

<https://web.ma.utexas.edu/users/benzvi/math/S-W>

REF singularity theory

#reference

singularity theory

- [V I Arnold.Singularity theory volume I](#)
- [Hendryk Zoladek - The monodromy group \(2006\).pdf](#)
- <https://www.maths.ed.ac.uk/~v1ranick/papers/greuel.pdf> ^[1]
- [John Milnor - Singular Points of Complex Hypersurfaces \(AM-61\), Volume 61-Princeton University Press \(2016\).pdf](#)
- [PCompanionMathematics.pdf > page=391](#)
- <https://math.berkeley.edu/~giventh/papers/arn.pdf>
- <http://www2.egr.uh.edu/~ychen11/PUBLICATIONS/BookChapter.pdf>
- <https://arxiv.org/pdf/2305.19842.pdf>
- <https://www.birs.ca/workshops/2012/12w5067/report12w5067.pdf>
- <https://www.sissa.it/fa/download/publications/remizov.pdf>
- [A singular mathematical promenade](#)

previous references

references

✓ https://chessapig.github.io/files/presentations/A_B_Models.pdf

- Arnold, Dynamical systems VI: Singularity theory, [V. I. Arnold, V. V. Goryunov, O. V. Lyashko, V. A. Vasil'ev - Singularity Theory I-Springer-Verlag Berlin Heidelberg \(1998\).pdf](#)
 - Math: Phenomenological account of singularity theory (few proofs). Reasonably quick to read, Arnold is good at writing.
 - <https://www.springer.com/gp/book/9783540637110>

- V.I. Arnold, A.N. Varchenko & S.M. Gusein-Zade, Singularities of Differentiable Maps Volume II: Monodromy and Asymptotic Integrals, 2012
 - Math: More detailed account of above book. Has section on asymptotic oscillating integrals.
 - <https://www.springer.com/gp/book/9781461284086>
- Greene and Vafa, Calabi-Yau Manifolds and Renormalization Group flows, 1989
 - Physics: **Describes CY mflds as RG fixed points of LG models**
 - [REF CYMan x RG flows](#)
- Vafa and Warner, Catastrophes and the classification of Conformal Theories, 1988
 - Physics: *Uses singularity theory to classify CFTs, derives ADE classification*
- Vafa, Topological Mirrors and Quantum Rings, 1991
 - *Physics for Math readers: Survey style summary of mirror symmetry*
- Griffiths, Periods of Integrals on Algebraic Manifolds, 1969
 - Math: *Survey of variations of hodge structures and period maps*
- Cecotti, Girardello, and Pasquinucci, Singularity Theory and N=2 Supersymmetry, 1990
 - Physics: *Describes dictionary between singularity theory and LG model physics*
- Witten, Mirror Manifolds and Topological Field Theory, 1991
 - Physics: *Notes on mirror symmetry in context of topological field theory*
- Barannikov, Quantum Periods 1: Semi-infinite variations of Hodge structures, 2000
 - Math: *Introduced VSHSs, On moduli space of noncommutative cmplx deformations*

descriptions

[NSF Award Search: Award # 1104329 - International Conference on Singularity Theory and Applications](#)

Singularity theory is a meeting place of many disparate areas of mathematics, where different types of ideas, techniques and results merge together. The modern theory of singularities dates back to the 1960s, with the pioneering work of Thom, Hironaka, Brieskorn, Zariski, and many other renowned mathematicians. More recently, *Singularity theory promoted vigorous interchanges among mathematical fields such as algebraic and geometric topology, algebraic geometry, number theory, and more applied fields such as the study of configurations in robot motion planning.*

In the last century, considerable effort has been directed towards studying manifolds - spaces that locally look uniform, at each point and in each direction. This effort has been immensely successful; a substantial part of our insight has been gained through the study of various invariants (e.g., characteristic numbers and classes), the surgery program, etc. In recent decades, *topologists have studied "singular" spaces with increasing interest, due to their numerous occurrences and applications within pure mathematics (algebraic geometry, number theory) and outside pure mathematics (mathematical physics, robot motion planning).* In contrast to a manifold, a singular space may locally look different from point to point. The study of topological properties of singular spaces developed into the field of Singularity theory. The proposed conference will focus around recent developments in this fast advancing field of research.

[Singularity theory - Isaac Newton Institute](#)

The main goal in most problems of singularity theory is to understand the dependence of some objects of analysis and geometry, or physics, or from some other science on parameters. For generic points in the parameter space their exact values influence only the quantitative aspects of the phenomena, their qualitative, topological features remaining stable under small changes of parameter values.

However, for certain exceptional values of the parameters these qualitative features may suddenly change under a small variation of the parameter. This change is called a perestroika, bifurcation or catastrophe in different branches of the sciences. A typical example is that of Morse surgery, describing the perestroika of the level variety of a function as the function crosses through a critical value. *(This has an important complex counterpart - the Picard-Lefschetz theory concerning the branching of integrals.)* Other familiar examples include caustics and outlines or profiles of surfaces obtained from viewing or projecting from a point, or in a given direction.

In spite of its fundamental character, and the central position it now occupies in mathematics, singularity theory is a surprisingly young subject. So, for example, one can consider the singularities arising from the orthogonal projections a generic surface in 3-space, a problem of surely classical interest. *Their classification was completed as recently as 1979. In one sense singularity theory can be viewed as the modern equivalent of the differential calculus, and this explains its central position and wide applicability.* In its current form the subject started with the fundamental discoveries of Whitney (1955), Thom (1958), Mather (1970), Brieskorn (1971). Substantial results and exciting new developments within the subject have continued to flow in the intervening years, while the theory has embodied more and more applications.

Newton Institute: Singularity theory and hyperbolicity, 2024

singularity theory and Bayesian statistics

Picard–Lefschetz theory, monodromy

[Picard–Lefschetz theory - Wikipedia](#)

[Monodromy - Wikipedia](#)

- analytic continuation and covering spaces

[Monodromy - Wikipedia](#)

🔗 [Henryk Żołądek - The Monodromy Group-Birkhäuser Basel \(2006\), p.7](#)

After publication of “Analysis situs” by H. Poincaré, investigation of the topology of algebraic varieties began.

representation theory

REF representation theory

#reference

representation theory

- [Fulton Harris Representation theory.pdf](#)
- [Peter Woit - Quantum Theory, Groups and Representations_ An Introduction\(2017\).pdf](#)
- [Modern Trends in Algebra and Representation Theory \(cambridge.org\)](#)

✓ <https://arxiv.org/pdf/0901.0827v5.pdf>



frame: PDF
style: height: 900px;
urlSuffix: arxiv.org/pdf/0901.0827v5.pdf

lecture videos

- [Programa de Doutorado: Lie Groups, Representation Theory and Symmetric Spaces - YouTube](#)
- [Representation Theory: A Combinatorial Viewpoint - YouTube](#)
- [M S Raghunathan.cpt Lie and reps](#)
- [Unitary Representations of Lie Groups | ETH Zürich Videoportal](#)
- [CMI Lie algebras - YouTube Anupam Singh - AIS on Lie Algebras \(4-23 July 2011\) CMI-IMSc, Chennai \(google.com\)](#)
- [Introduction to Lie Algebras Doutorado IMPA Verao 2011 Reimundo Heluani 03/01/2011 aula 01 parte1 \(youtube.com\)](#)
- [IMPA Meeting: Quantum Groups and 3-Manifold Invariants - YouTube](#)

usual and beyond

- <https://math.mit.edu/~etingof/reprbook.pdf>
- https://ocw.mit.edu/courses/18-755-lie-groups-and-lie-algebras-ii-spring-2024/mit18_755_s24_lec_full.pdf
- [Lecture notes on Cherednik Algebras | Double Affine Hecke Algebras in Representation Theory, Combinatorics, Geometry, and Mathematical Physics | Mathematics | MIT OpenCourseWare](#)
- https://www.mat.uniroma2.it/~damiani/RepresentationTheory_Etingof.pdf
- [moduli of quiver reps and VBs](#)

geometric representation theory

[geometric representation theory in nLab \(ncatlab.org\)](#)

Representation theory is the study of the basic symmetries of mathematics and physics. Symmetry groups come in many different flavors: finite groups, Lie groups, p-adic groups, loop groups, adelic groups,... A striking feature of representation theory is the persistence of fundamental structures and unifying themes throughout this great diversity of settings. One such theme is the Langlands philosophy, a vast nonabelian generalization of the Fourier transform of classical harmonic analysis, which serves as a visionary roadmap for the subject and places it at the heart of number theory.

The fundamental aims of geometric representation theory are to uncover the deeper geometric and categorical structures underlying the familiar objects of representation theory and harmonic analysis, and to apply the resulting insights to the resolution of classical problems. *A groundbreaking example of its success is Beilinson-Bernstein's generalization of the Borel-Weil-Bott theorem, giving a uniform construction of all representations of Lie groups via the geometric study of differential equations on flag varieties.*

The geometric study of representations often reveals deeper layers of structure in the form of categorification. Categorification typically replaces numbers (such as character values) by vector spaces (typically cohomology groups), and vector spaces (such as representation rings) by categories (typically of sheaves). It is a primary explanation for miraculous integrality and positivity properties in algebraic combinatorics. A recent triumph of geometric methods is Ngô's proof of the Fundamental Lemma, a key technical ingredient in the Langlands program. The proof relies on the cohomological interpretation of orbital integrals, which makes available the deep topological tools of algebraic geometry (such as Hodge theory and the Weil conjectures).

- [Geometric Representation Theory and Gauge Theory: Cetraro, Italy 2018 | SpringerLink](#) **STUDYING** [gauge theory](#)
- [Raphaël Rouquier - Geometric representation theory as representation-theoretic geometry \(youtube.com\)](#)
- [QGpublic.pdf \(ed.ac.uk\)](#)
- [Lecture 1 | Geometric representation theory \(youtube.com\)](#)
- [Representation theory as gauge theory - Clay Mathematics Institute](#)
- [2021 IHES Summer School - Enumerative Geometry, Physics and Representation Theory](#)
- [A.Okounkov's lectures, math.columbia.edu/~okounkov/33lectures.pdf](#)
- [Math 569: Flag Varieties](#)

- [Elliot Kienzle](#) [Understanding hamiltonian G-spaces through quantization | Elliot Kienzle \(chessapig.github.io\)](#)
- [HIM Lectures: Trimester Program "Symplectic Geometry and Representation Theory" - YouTube](#)

✓ https://escholarship.org/content/qt07t5s31b/qt07t5s31b_noSplash_b004dfc53a64fff1ad5536bb1e3849d3.pdf

- [Representation Theory and Complex Geometry | SpringerLink](#)
- [Hodge Theory, Complex Geometry, and Representation Theory \(ams.org\)](#)

integrable systems

REF [representation theory](#) and integrable systems

- [Representation Theory, Mathematical Physics, and Integrable Systems: In Honor of Nicolai Reshetikhin | SpringerLink](#)
- [Integrable Systems, Spectral Curves and Representation Theory A. Lesfari](#)
- [Leonid Rybnikov](#) Representation Theory, Integrable Systems, Quantization, Kashiwara Crystals
- [Ivan Motorin](#) Cluster Algebras, Resolution of Singularities, Representation Theory, Integrable Systems
- [Representation Theory and Integrable Systems \(iupui.edu\)](#)
- [Representation theory, gauge theory, and integrable systems \(3-8 February 2019\): Scientific Programme · Kavli IPMU Indico System \(Indico\)](#)

MSRI summer school 2014: Complex geometry and geometric analysis on complex manifolds

💡 Riemann Surfaces by Jacob Bernstein (bernstein@math.jhu.edu)

https://www.bilibili.com/video/BV1fW41197nr/?spm_id_from=333.337.search-card.all.click

For MSRI summer school 2014: Complex geometry and geometric analysis on complex manifolds

- **Prerequisites:**

- Knowledge of basic complex analysis—at the level of Ahlfors, Complex Analysis, Chapters 1-5—will be assumed. Some basic familiarity with (abstract) surface theory and differential forms will be helpful. However, I will review this material as needed.

- **Reading:**

- The main text will be Donaldson, Riemann Surfaces; get at <http://www2.imperial.ac.uk/~skdona/RSPREF.PDF>.
- Syllabus: https://www.slmath.org/ckeditor_assets/attachments/106/bernstein_hein_naber_syllabus.pdf

- **Other useful references:**

- Farkas and Kra, Riemann Surfaces; a classical text on the subject.
- Miranda, Algebraic Curves and Riemann Surfaces; a more algebraic perspective.

- **Week 1: Introduction to Riemann Surfaces**

- Surfaces and Topology
- Riemann Surfaces and Holomorphic Maps
- Maps between Riemann Surfaces
- Calculus on Riemann Surfaces
- De Rham Cohomology

- **Week 2: Geometric Analysis on Riemann Surfaces**

- Elliptic Functions and Integrals
- Meromorphic Functions
- Inverting the Laplacian
- The Uniformization Theorem
- Riemann Surfaces and Minimal Surfaces

💡 Geometric Analysis (Aaron Naber, anaber@math.northwestern.edu)

https://www.bilibili.com/video/BV1fW41197nr/?spm_id_from=333.337.search-card.all.click

- **Prerequisites:** Basics of manifolds, tensors, and differential forms. Basics of pde theory, for instance Evans's book Partial Differential Equations, in particular those chapters on second order elliptic and parabolic equations. Familiarity with exponential maps, injectivity radius, and geodesics would be helpful, for instance chapter one of Jost's book Riemannian Geometry and Geometric Analysis is more than sufficient.
- **Reading:** The main source will be Petersen's book on Riemannian Geometry. We will also rely on Jost's Riemannian Geometry and Geometric Analysis, and on the book by Cheeger Degeneration of Riemannian Metrics Under Ricci Curvature Bounds. More advanced topics will use relevant papers in the field.
- **Week 1: Introduction to Geometric Analysis**
 - Review of Manifolds and Smooth Structure
 - Introduction to Curvature and Geodesic Coordinates

- Laplacians and Harmonic Coordinates
- Heat Kernels and Geometry
- Sectional Curvature and Finite Diffeomorphism Theorems
- Week 2: Topics in Regularity Theory
 - Ricci Curvature, Volume Monotonicity and Rigidity Theorems
 - Ricci Curvature and Almost Rigidity Theorems
 - Lower Ricci Curvature and Stratification Theorems
 - Bounded Ricci Curvature and ε -regularity Theorems
 - Outline of Regularity Theory for Einstein Manifolds

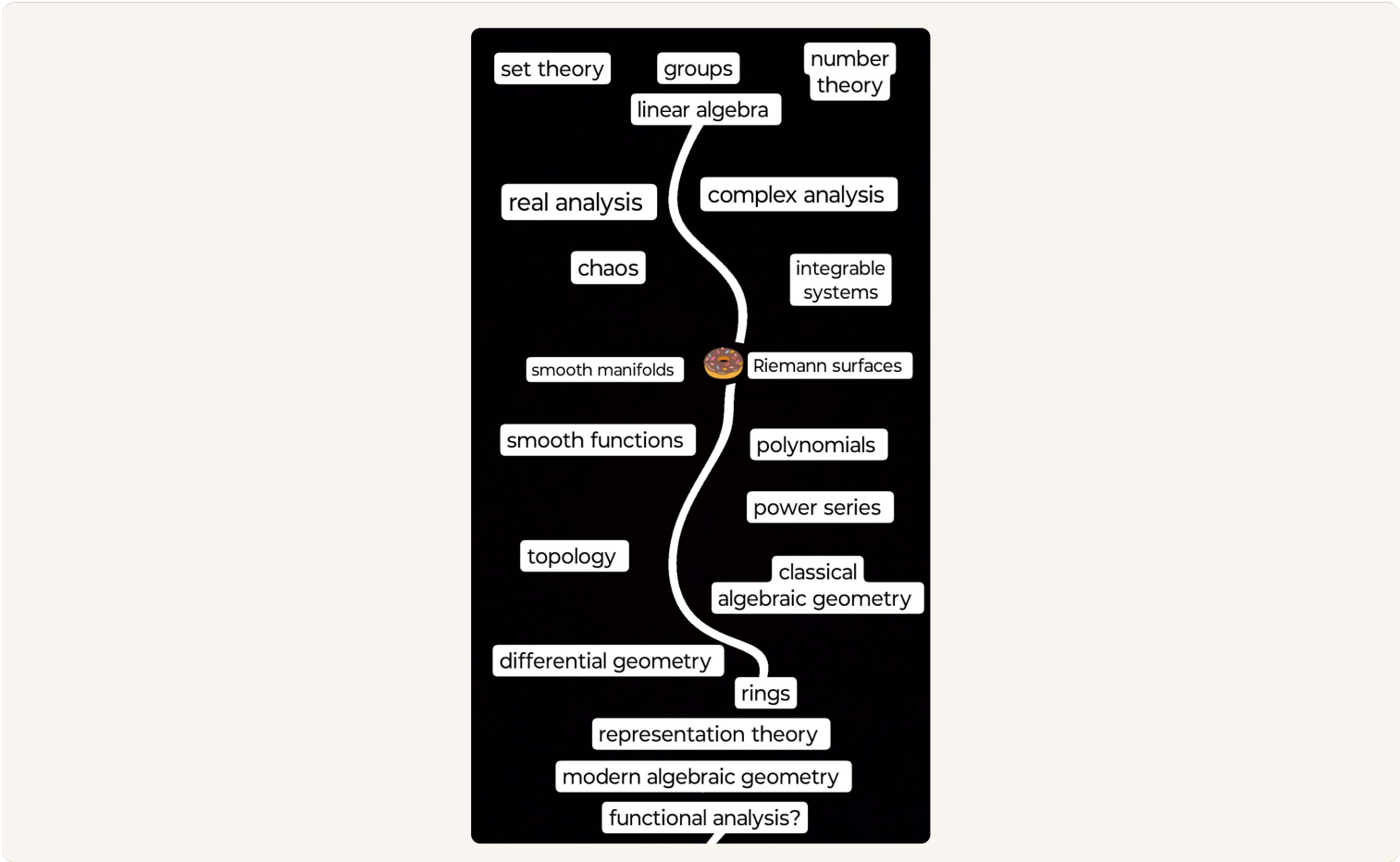
💡 Complex Geometry (Hans-Joachim Hein, hansjoachim.hein@univ-nantes.fr)

https://www.bilibili.com/video/BV1Qv41177Hp/?spm_id_from=333.337.search-card.all.click

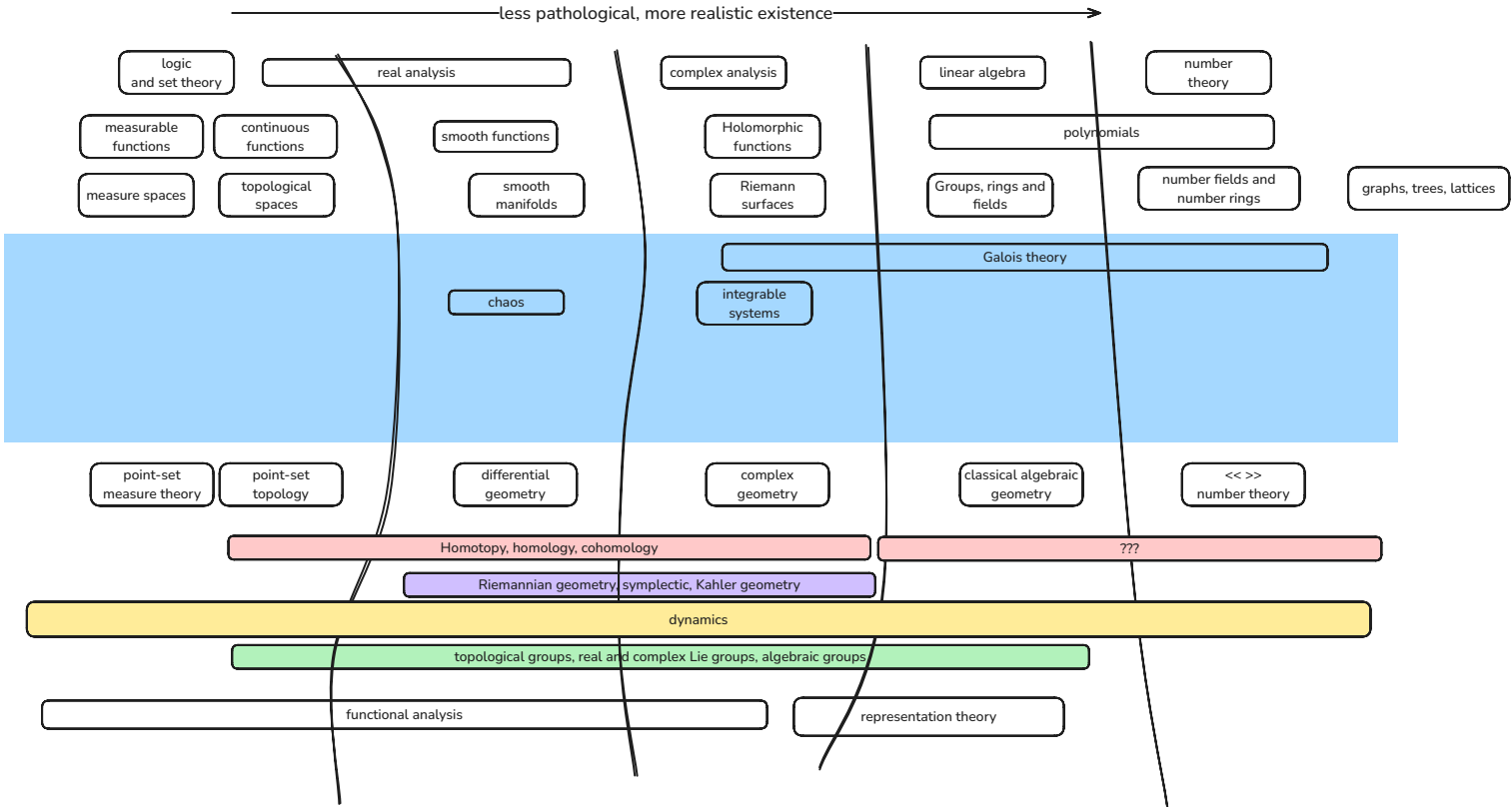
- Prerequisites: Basics of manifolds, tensor fields, differential forms, etc. Warner, Foundations of Differentiable Manifolds and Lie Groups, Chapters 1, 2, 4, 6, contains all we need and much more.
- Basic complex analysis as in Stein & Shakarchi, Complex Analysis, Chapters 1, 2, 3, 8.
- Reading:
 - Huybrechts, Complex Geometry, is an excellent basic textbook with exercises.
 - Lecture notes by Joel Fine: <http://homepages.ulb.ac.be/~joelfine/papers.html#survey>.
 - Complex Monge-Ampere: <http://gamma.im.uj.edu.pl/~blocki/publ/ln/tln.pdf>.
 - For the end of Week 2: <http://arxiv.org/pdf/0803.0985.pdf>, Section 5.
- Week 1: Introduction to Complex Geometry
 - Holomorphic Functions and Complex Calculus
 - Complex Manifolds
 - Holomorphic Line Bundles
 - Pseudoconvexity and Pseudoconcavity
 - The Kodaira Embedding Theorem
- Week 2: Topics in Kahler-Einstein Manifolds
 - Kahler Manifolds
 - Ricci Curvature and the Complex Monge-Ampere Equation
 - Examples of Ricci-flat Spaces
 - Basic Estimates for the Complex Monge-Ampere Equation
 - The Mukai-Umemura Manifold

philosophy of mathematics, 16 march '25 edition

Exactly one year ago, I made the following distinction



I want to improve on this art:



sett.inside.dualities

Dualities of structures

objects $\leftrightarrow$ morphisms

space $\leftrightarrow$ functions on it: between algebra and geometry

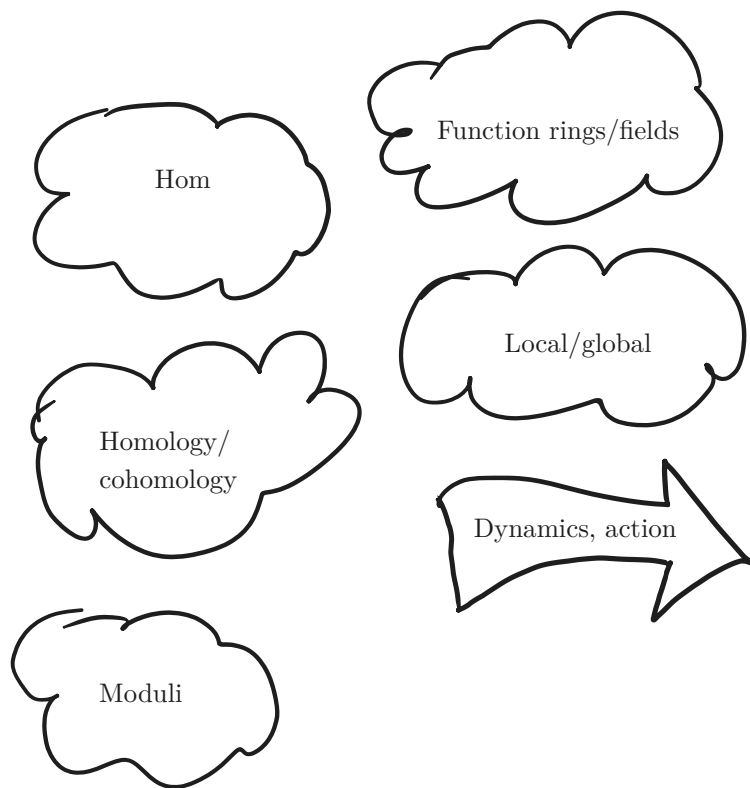
[1] [2]

	space (<i>geometry</i>)	<i>algebra</i> of functions on the space
linear algebra	finite dimensional vector space V FinVec_k	dual of V $\text{FinVec}_k^{\text{op}}$
topology	$\text{Top}_{\text{H,cpt}}$	$\hookrightarrow \text{CAlg}^{\text{op}}$ but $\sim \text{TopCAlg}_{\mathbb{C}*}^{\text{op}}$
differential geometry	Man	$\hookrightarrow \text{CAlg}_{\mathbb{R}}^{\text{op}}$
algebraic geometry	$\text{Schemes}_{\text{Aff}}$	$\text{FinCAlg}^{\text{op}}$
supergeometry		
noncomm topology		
$\dim = 1$ complex geometry	compact complex 1-manifolds	field extensions of $\mathbb{C}(z)$

dual objects

- projective space $\leftrightarrow$ dual projective space
- lattice $\leftrightarrow$ dual lattice
- group $\leftrightarrow$ dual group
- ...

1. [duality between algebra and geometry in nLab \(ncatlab.org\)](#) $\leftrightarrow$
2. [Manifolds](#) $\leftrightarrow$



3: 2-categories

